EC8451. ELECTRO MAGNETIC FIELDS.

UNIT I : INTRODUCTION

Electromagnetic fields are combination of invisible electric and magnetic fields of force. They are caused by electric charges at rest and in motion. Positive and negative electric charges are sources of the electric fields and moving electric charges yielding a current is the source of magnetic fields. Time varying electric and magnetic fields are coupled in an electromagnetic field radiating from the source.

Electromagnetic Model:

* Model - describes how devices or objects of interest behave (ie) "a representation of the behaviour of real devices and objects is called model".

* Electromagnetic model - process of modeling the interaction of EM fields with physical objects and the environment.

- It is used to design and model of antenna, radar, satellite, medical and other applications.

Blocks used to model an EM modeling are

* Magnetic Elements
* Magnetic Sensors
* Magnetic Sources.

1. Magnetic Elements :

* Electromagnetic converter :- Lossless electro magnetic energy conversion device provides a general interface between the electrical and magnetic domains. (fig.a)

* Fundamental reluctance :- provides simplified model of magnetic reluctance. (ie) a component that resists flux flow. (fig.b)

* Magnetic reference :- represents a reference point for all magnetic conserving ports. (fig.c)

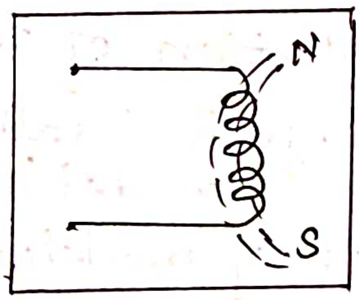


fig. (a)

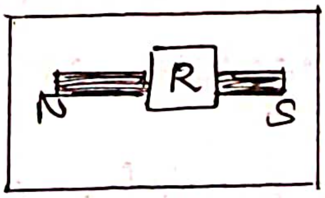


fig. (b)

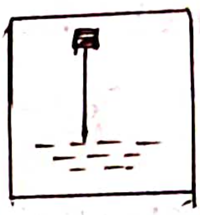


fig. (c)

2. Magnetic Sensors :

* Flux Sensor :- converts flux measured in any magnetic branch into physical signal proportional to the flux. (fig.d)

* MMF Sensor: - converts the mmf measured between any magnetic connections into physical signal proportional to mmf. (fig. e)

mmf - magneto motive force.

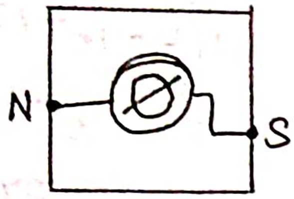


fig. (d)

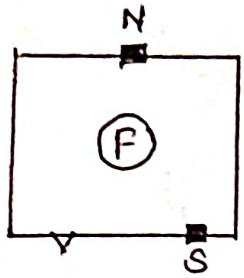


fig. (e)

3) Magnetic Sources:

* Controlled Flux Source: - Used to maintain the specified flux through it. (fig. f)
* Controlled mmf Source: - used to maintain specified mmf at its output. (fig. g)

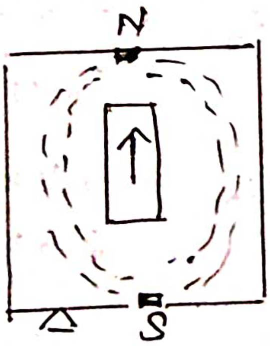


fig. (f)

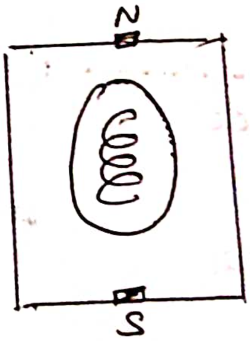


fig. (g)

Units and Constants:

Constants: - Physical quantity or parameter used to express a relation or property which remains the same in all circumstances.

Units :- Standard for measurement of physical quantities or parameters

Constants/ Physical Quantity	Units	Symbol	Approximate Value
Electron charge	C	e	-1.6×10^{-19}
Electron mass	Kg	m_e	9.1×10^{-31}
Proton mass	Kg	m_p	1.67×10^{-27}
Neutron mass	Kg	m_n	1.67×10^{-27}
Electron volt	J	eV	1.6×10^{-19}
Acceleration due to gravity	m/s^2	g	9.8
Universal constant of gravitation	$m^2/kg\,s^2$	G	6.66×10^{-11}
Permittivity of free space	F/m	ϵ_0	8.854×10^{-12} (or) $\frac{1}{36\pi} \times 10^{-9}$
Permeability of free space	H/m	μ_0	$4\pi \times 10^{-7}$ (or) 12.6×10^{-7}
Intrinsic impedance of free space	Ω	η_0	376.6 or 120π
Speed of light in vaccum	m/s	c	3×10^8
Planck's constant	Js	h	6.62×10^{-34}
Boltzmann constant	J/k	k	1.38×10^{-23}

C → Coulomb | m → meter | H → Henry
Kg → Kilogram | s → Second | Ω → Ohm
J → Joule | F → Farad | K → Kelvin.

Review of Vector Algebra :

Introduction :

Electromagnetics is the branch of electrical engineering in which the electric and magnetic phenomena are studied. For most electromagnetic problems, there are three space variables and thus the solutions become complex. This can be overcome by the use of vector analysis. The use of vector analysis in the study of electromagnetic field theory results in less time for solutions.

Any physical quantity can be represented either as a Scalar or a vector.

Scalar : A quantity that has only magnitude is said to be Scalar quantity.

Ex : Time, mass, distance, temperature etc.

Vector : A quantity that has both magnitude and direction is called vector quantity.

Ex : Force, velocity, displacement etc.

Vector quantity is represented by $\vec{A}$, $\vec{B}$.

Fields : If at each point of a region, there is a value of some physical function, the region is called field.

Fields maybe classified as Scalar fields and vector fields.

Scalar field: If the value of physical function at each point is a Scalar quantity, then the field is scalar field.

Ex: Temperature distribution in a building.

Vector Field: If the value of the physical function at each point is a vector quantity, then the field is known as vector field.

Ex: Gravitational force on a body in Space.

Vector Algebra:

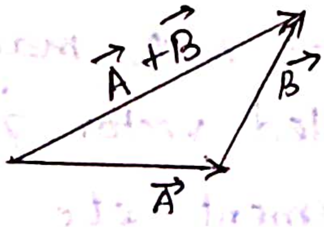
Vector Addition and Subtraction:

Let $\vec{A}$ and $\vec{B}$ be two vectors given as

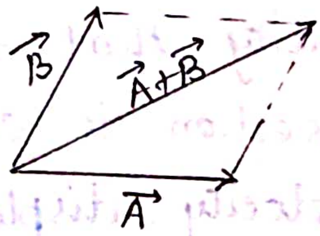
$$\vec{A} = A_x \bar{a}_x + A_y \bar{a}_y + A_z \bar{a}_z \; , \; \vec{B} = B_x \bar{a}_x + B_y \bar{a}_y + B_z \bar{a}_z$$

then, Sum $\vec{A}+\vec{B} = (A_x+B_x)\bar{a}_x + (A_y+B_y)\bar{a}_y + (A_z+B_z)\bar{a}_z$

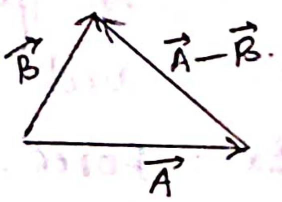
Difference $\vec{A}-\vec{B} = (A_x-B_x)\bar{a}_x + (A_y-B_y)\bar{a}_y + (A_z-B_z)\bar{a}_z$



a) Triangle rule addition



b) Parallelogram rule addition



c) Subtraction.

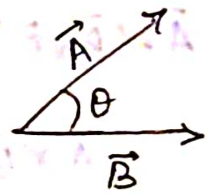
Multiplication: When a vector is multiplied by a Scalar whose direction is same as original and magnitude is product of magnitudes of vector + Scalar.

$\vec{A} = a\vec{B}$; $\vec{A}$ - New vector, $\vec{B}$ - Given vector,
a - given Scalar. $\xrightarrow{\vec{B}}$ $\xrightarrow{a\vec{B}}$

When two vectors are multiplied, the result maybe scalar or vector depending on how they are multiplied. There are two types:

* Scalar Product (Dot Product):

Scalar multiplication of two vectors is a scalar quantity whose magnitude is the product of magnitudes of vectors multiplied by cosine of the angle between them.



Dot product $\vec{A}\cdot\vec{B} = AB\cos\theta$.

It obeys commutative law $\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$

Let $\vec{A} = A_x\bar{a}_x + A_y\bar{a}_y + A_z\bar{a}_z$ & $\vec{B} = B_x\bar{a}_x + B_y\bar{a}_y + B_z\bar{a}_z$

$$\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z.$$

$$\because \bar{a}_x\cdot\bar{a}_x = \bar{a}_y\cdot\bar{a}_y = \bar{a}_z\cdot\bar{a}_z = 1$$

$$\bar{a}_x\cdot\bar{a}_y = \bar{a}_y\cdot\bar{a}_z = \bar{a}_z\cdot\bar{a}_x = 0$$

* Vector Product (Cross Product):

Vector Product of two vectors is a vector whose magnitude is the product of magnitudes of two vectors and sine of the angle between them

Cross Product $\vec{A}\times\vec{B} = AB\sin\theta\,\hat{n}$;

$\hat{n}$ - unit vector normal to the plane contains $\vec{A}$ & $\vec{B}$

$$\vec{A}\times\vec{B} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} ; \qquad \vec{B}\times\vec{A} = -\vec{A}\times\vec{B}.$$

The condition for two vectors A & B to be parallel is given by $\bar{A} \times \bar{B} = 0$

The condition for two vectors to be perpendicular is given by $\bar{A} \cdot \bar{B} = 0$

Problem 1 : Given that $\bar{A} = \bar{a}_x + \bar{a}_y$, $\bar{B} = \bar{a}_x + 2\bar{a}_z$ and $\bar{C} = 2\bar{a}_y + \bar{a}_z$. Find $\bar{A} \cdot (\bar{B} \times \bar{C})$ & $(\bar{A} \times \bar{B}) \cdot \bar{C}$

$$\bar{B} \times \bar{C} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ 1 & 0 & 2 \\ 0 & 2 & 1 \end{vmatrix} = -4\bar{a}_x - \bar{a}_y + 2\bar{a}_z$$

$$\bar{A} \cdot (\bar{B} \times \bar{C}) = (\bar{a}_x + \bar{a}_y) \cdot (-4\bar{a}_x - \bar{a}_y + 2\bar{a}_z) = -4 - 1 = -5$$

$$\bar{A} \times \bar{B} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ 1 & 1 & 0 \\ 1 & 0 & 2 \end{vmatrix} = 2\bar{a}_x - 2\bar{a}_y - \bar{a}_z$$

$$(\bar{A} \times \bar{B}) \cdot \bar{C} = (2\bar{a}_x - 2\bar{a}_y - \bar{a}_z) \cdot (2\bar{a}_y + \bar{a}_z) = -4 - 1 = -5$$

$$\bar{A} \cdot (\bar{B} \times \bar{C}) = (\bar{A} \times \bar{B}) \cdot \bar{C}$$

Problem 2 : Show that the vectors $\bar{A} = 4\bar{a}_x - 2\bar{a}_y + 2\bar{a}_z$ and $\bar{B} = -6\bar{a}_x + 3\bar{a}_y - 3\bar{a}_z$ are parallel to each other.

$$\bar{A} \times \bar{B} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ 4 & -2 & 2 \\ -6 & 3 & -3 \end{vmatrix} = \bar{a}_x (6-6) - \bar{a}_y (-12+12) + \bar{a}_z (12-12)$$

$$= 0.$$

$\bar{A} \times \bar{B} = 0$ then $\bar{A}$ & $\bar{B}$ are parallel to eachother

Problem 3 : Show that $\bar{A} = 6\bar{a}_x + 2\bar{a}_y - 5\bar{a}_z$ & $\bar{B} = 5\bar{a}_x - 5\bar{a}_y + 4\bar{a}_z$ are perpendicular to each other.

$$\bar{A} \cdot \bar{B} = (6 \times 5) + (2 \times (-5)) - (5 \times 4) = 0$$

$\bar{A} \cdot \bar{B} = 0$, then $\bar{A}$ & $\bar{B}$ are perpendicular to eachother.

Co-ordinate Systems:

It is defined as a system to describe uniquely the spatial variation of a quantity at all points in space.

* Orthogonal Coordinate System:
  Three coordinate axes are perpendicular to each other.

* Non-Orthogonal Co-ordinate System:
  Co-ordinate axes are not perpendicular to each other.

The orthogonal co-ordinate systems are

* Cartesian or Rectangular Co-ordinates
* Cylindrical Coordinates
* Spherical coordinates.

1) Cartesian Coordinate System: (Rectangular Co-ordinate)

In this, three co-ordinate axes x, y & z are mutually right angles to each other.
Consider a point $P(x,y,z)$ in space at a distance 'r' from the origin.

$r = x\bar{a}_x + y\bar{a}_y + z\bar{a}_z$ where $\bar{a}_x, \bar{a}_y$ & $\bar{a}_z$ are unit vectors.

A unit vector in a given direction is a vector in that direction divided by its magnitude

(ie) $$\bar{a}_r = \frac{\bar{r}}{|r|} = \frac{x\bar{a}_x + y\bar{a}_y + z\bar{a}_z}{\sqrt{x^2+y^2+z^2}}$$

Consider the points $P(x,y,z)$ & $Q(x+dx, y+dy, z+dz)$.

Fig.1 shows six planes define a rectangular parallel piped.

The differential length dl from P to Q is the diagonal of the parallel piped is given by $dl = \sqrt{(dx)^2+(dy)^2+(dz)^2}$

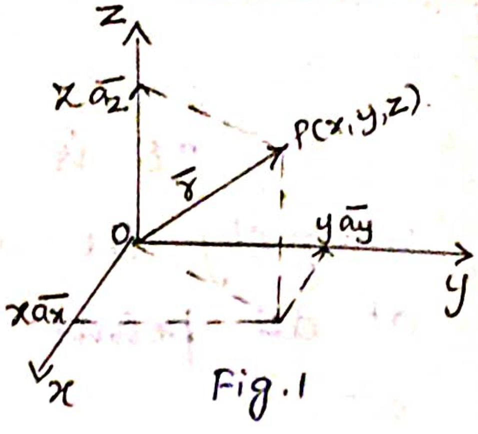


Fig.1

The differential area $ds = dx\,dy = dy\,dz = dz\,dx$

The differential volume $dv = dx\,dy\,dz$.

2) Cylindrical Co-ordinate System:

It is three dimensional version of polar coordinates of analytic geometry. Consider any point as the intersection of three mutually perpendicular surfaces. They are circular cylinder (ρ = constant) plane (ϕ = constant) and another plane (z = const). The co-ordinates are ρ, ϕ and z.

Differential length $dl = \sqrt{(d\rho)^2+(\rho d\phi)^2+(dz)^2}$

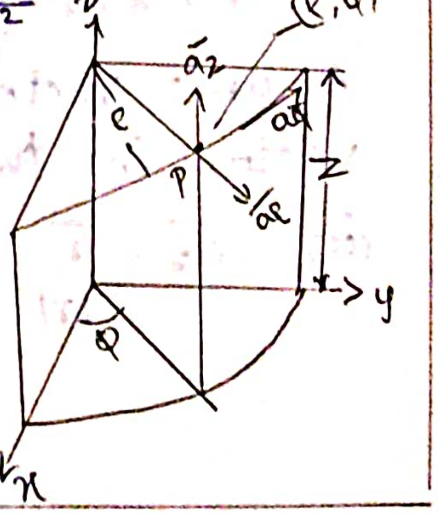
Differential area $ds = \rho\, d\rho\, d\phi$

$= d\rho\, dz$

$= \rho\, d\phi\, dz$

Differential volume $dv = d\rho\, \rho d\phi\, dz$

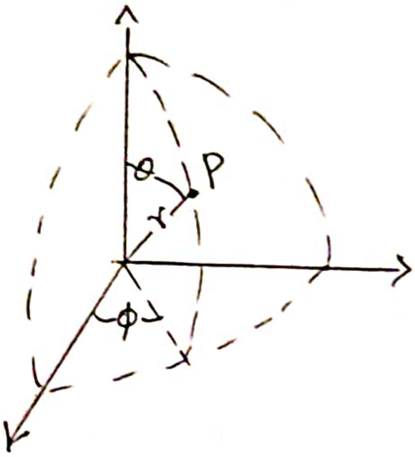
$= \rho\, d\rho\, d\phi\, dz$



3) Spherical Co-ordinate System:

Consider any point as the point of intersection of the spherical surface (radius r = constant) conical surface (θ, angle between r & z = constant) and plane surface (ϕ = constant). Co-ordinates of this system are r, θ, ϕ.

A differential volume element maybe obtained increasing r, θ, ϕ by $dr, d\theta, d\phi$.



The differential length

$$dl = \sqrt{dr^2 + (r d\theta)^2 + (r \sin\theta \, d\phi)^2}$$

The differential area $ds = r dr \, d\theta$

$$= dr \cdot r\sin\theta \, d\phi = r \sin\theta \, d\phi \, dr$$

$$= r d\theta \cdot r\sin\theta \, d\phi = r^2 \sin\theta \, d\theta \, d\phi.$$

The differential volume $dv = dr \cdot r d\theta \cdot r\sin\theta \, d\phi$

$$= r^2 \sin\theta \, d\theta \, d\phi \, dr$$

Transformation:

Cartesian to cylindrical: $\rho = \sqrt{x^2+y^2}$; $\phi = \tan^{-1} y/x$; $z = z$

Cylindrical to cartesian: $x = \rho\cos\phi$; $y = \rho\sin\phi$; $z = z$

Cartesian to spherical: $r = \sqrt{x^2+y^2+z^2}$; $\theta = \cos^{-1}(z/r)$;
$\phi = \tan^{-1}(y/x)$.

Spherical to Cartesian: $x = r\sin\theta\cos\phi$; $y = r\sin\theta \cdot \sin\phi$
$z = r\cos\theta$.

Ex 1: Transform cartesian co-ordinates $x=2, y=1, z=3$ into spherical co-ordinates.

$r = \sqrt{x^2+y^2+z^2} = \sqrt{4+1+9} = 3.74$; $\theta = \cos^{-1}(z/r) = \cos^{-1}\left(\frac{3}{\sqrt{14}}\right) = 36.7^\circ$

$\phi = \tan^{-1}(y/x) = \tan^{-1}(1/2) = 26.56$

Divergence Theorem:

The volume integral of the divergence of a vector field over a volume is equal to the surface integral of the normal component of this vector over the surface bounding the volume

$$\iiint_V \nabla \cdot A \, dv = \oint\!\!\oint_S A \cdot ds$$

Proof: $\nabla \cdot A = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$

Take volume integral on both sides,

$$\iiint_V \nabla \cdot A \, dv = \iiint_V \left[\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right] dx\,dy\,dz \quad (\because dv = dx\,dy\,dz)$$

Consider an elemental volume,

$$\iiint_V \frac{\partial A_x}{\partial x} dx\,dy\,dz = \iint \left[\int \frac{\partial A_x}{\partial x} dx\right] dy\,dz, \text{ but } \int_{x_1}^{x_2} \frac{\partial A_x}{\partial x} dx = A_{x_2} - A_{x_1} = A_x$$

where, A_{x_1}, A_{x_2} are x-components of vector on left & right side of origin along x-axis.

$$\iiint_V \frac{\partial A_x}{\partial x} dx\,dy\,dz = \iint A_x \, dy\,dz = \iint_S A_x \, ds_x .$$

where, $dy\,dz = ds_x \Rightarrow x$ component of surface area ds.

Similarly, $\iiint_V \frac{\partial A_y}{\partial y} dx\,dy\,dz = \iint_S A_y \, ds_y .$

$$\iiint_V \frac{\partial A_z}{\partial z} dx\,dy\,dz = \iint_S A_z \, ds_z .$$

Then, $\iiint_V \nabla \cdot A \, dv = \iiint_V \left[\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right] dx\,dy\,dz$

$$= \iint_S (A_x ds_x + A_y ds_y + A_z ds_z) = \oint\!\!\oint_S A \cdot ds$$

$$\therefore \iiint_V \nabla \cdot A \, dv = \oint\!\!\oint_S A \cdot ds$$

Line, Surface and Volume Integrals:

In electromagnetic field, a charge can exist in point form, line form, Surface form and volume form. For analysing those charge distribution, various types of integrals are used; They are,

i) Line integral
ii) Surface integral
iii) Volume integral

i) Line Integral:

* If a charge is uniformly distributed along a line, then it is called line charge. It may be a straight line or a distance travelled along a curve.

* The corresponding line integral is given by,

$$\int F.dl = \int |F| \cos\theta \, dl.$$

* The line integral is also called contour integral, and its direction is always assumed to be positive.

ii) Surface Integral :

* If a charge is distributed over a two dimensional surface, then it is called surface charge. The charge per unit area is called surface charge density → "ρ_s".

* The corresponding surface integral (or) flux of $\bar{F}$ through surface 's' is given by,

$$\phi = \int_S F \cdot ds = \int_S |\bar{F}| \cos\theta \, ds .$$

iii) Volume Integral :

* If the charge distribution exists in a three dimensional volume, it is called as volume charge. The charge per unit volume is called as volume charge density "ρ_v".

* If ρ_v is the volume charge density over volume V, then the volume integral is given by.

$$\phi = \oint_V \rho_v \, dv .$$

Vector differential Operator:

The differential vector operator ∇ is called del or nabla defined as

$$\nabla = \frac{\partial}{\partial x}\bar{a}_x + \frac{\partial}{\partial y}\bar{a}_y + \frac{\partial}{\partial z}\bar{a}_z$$

There are three possible operations with Δ.

1. Gradient: The Gradient of any scalar function is the maximum space rate of change of that function. If V represent electric potential, then ∇V represents potential gradient.

$$\nabla V = \frac{\partial V}{\partial x}\bar{a}_x + \frac{\partial V}{\partial y}\bar{a}_y + \frac{\partial V}{\partial z}\bar{a}_z$$

$$\nabla V = \text{grad } V.$$

Gradient of scalar is a vector.

2. Divergence: The divergence of a vector 'A' at any point is defined as limit of its surface integrated per unit volume as the volume enclosed by the surface shrinks to zero.

$$\nabla \cdot A = \lim_{V \to 0} \frac{1}{V} \oiint_S A \cdot \hat{n}\, ds$$

$$\nabla \cdot A = \left(\bar{a}_x \frac{\partial}{\partial x} + \bar{a}_y \frac{\partial}{\partial y} + \bar{a}_z \frac{\partial}{\partial z}\right)\left(\bar{a}_x A_x + \bar{a}_y A_y + \bar{a}_z A_z\right)$$

$$\nabla \cdot A = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla \cdot A = \text{div } A.$$

Divergence of a vector is Scalar.

3. Curl: The curl of a vector 'A' at any point is defined as limit of its surface integral of its cross product with normal over a closed surface per unit volume as the volume shrinks to zero.

$$|\text{curl } A| = \lim_{V \to 0} \frac{1}{V} \oiint_S \bar{n} \times A \, ds$$

$$\nabla \times A = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\bar{a_x} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\bar{a_y} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\bar{a_z}$$

$$\nabla \times A = \begin{vmatrix} \bar{a_x} & \bar{a_y} & \bar{a_z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_x & A_y & A_z \end{vmatrix}$$

. This operation is called curl.

$$\nabla \times A = \text{curl } A.$$

If $\nabla \cdot \bar{A} = 0$ (ie) divergence of $\bar{A}$ is zero, then it is said to be Solenoidal.

If $\nabla \times \bar{A} = 0$, (ie) curl of $\bar{A}$ is zero, then it is said to be irrotational.

Ex 1: Show that $\bar{A} = 3y^4z^2\bar{a_x} + 4x^3z^2\bar{a_y} + 3x^2y^2\bar{a_z}$ is solenoidal.

$$\nabla \cdot \bar{A} = \frac{\partial}{\partial x}(3y^4z^2) + \frac{\partial}{\partial y}(4z^2x^3) + \frac{\partial}{\partial z}(3x^2y^2)$$

$$= 0 + 0 + 0 = 0.$$

$\nabla \cdot \bar{A} = 0 \quad \therefore \bar{A}$ is Solenoidal.

Ex 2: Show that $\bar{F} = 2xy\,\bar{a_x} + (x^2 + 2yz)\bar{a_y} + (y^2+1)\bar{a_z})$ is irrotational.

$$\nabla \times \bar{F} = \begin{vmatrix} \bar{a_x}\,i & \bar{a_y}\,j & \bar{a_z}\,k \\ \frac{\partial}{\partial x} & \partial/\partial y & \partial/\partial z \\ 2xy & x^2+2yz & y^2+1 \end{vmatrix}$$

$$= \bar{a_x}\left[\frac{\partial}{\partial y}(y^2+1) - \frac{\partial}{\partial z}(x^2+2yz)\right] - \bar{a_y}\left[\frac{\partial}{\partial x}(y^2+1) - \frac{\partial}{\partial z}(2xy)\right] + \bar{a_z}\left[\frac{\partial}{\partial x}(x^2+2yz) - \partial/\partial y\,(2xy)\right]$$

$$= \bar{a_x}(2y-2y) - \bar{a_y}(0-0) + \bar{a_z}(2x-2x) = 0.$$

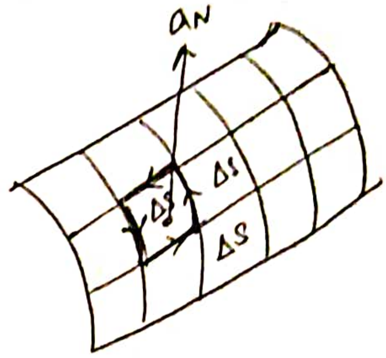
$\nabla \times \bar{F} = 0$, then $\bar{F}$ is irrotational.

Stoke's Theorem:

The line integral of a vector around a closed path is equal to the surface integral of the normal component of its curl over any closed surface.

$$\oint_c \bar{H}.dl = \iint_s (\nabla \times \bar{H}).ds.$$

Proof: consider an arbitrary surface this is broken up into incremental surfaces of area Δs.



If H is any field vector, then by definition of curl to one of these incremental surfaces.

$$\frac{\oint H.dl\ \Delta s}{\Delta s} = (\nabla \times H)_N$$

where N indicates normal to the surface.

$dl\ \Delta s \rightarrow$ closed path of an incremental area Δs.

The curl of H normal to the surface canbe written as

$$\frac{\oint H.dl\ \Delta s}{\Delta s} = (\nabla \times H).a_N$$

$$\oint H.dl\ \Delta s = (\nabla \times H).a_N \Delta s$$
$$= (\nabla \times H).\Delta s$$

where a_N is a unit vector normal to Δs.

The closed integral for whole surface s is given by the surface integral of the normal component of curl H.

$$\oint H.dl = \iint_s \nabla \times H.\ ds$$

Ex.1: Using divergence theorem, evaluate $\iint_S \bar{F}.\hat{n}\, ds$ where $\bar{F} = 2xy\,\bar{i} + y^2\bar{j} + 4yz\,\bar{k}$ and s is the surface of cube bounded by $x=0, x=1$; $y=0, y=1$; $z=0, z=1$.

By divergence theorem, $\iint_S \bar{F}.n\, ds = \iiint_V \nabla.\bar{F}\, dv$

$$\nabla.\bar{F} = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right).(2xy\,\bar{i} + y^2\bar{j} + 4yz\,\bar{k})$$

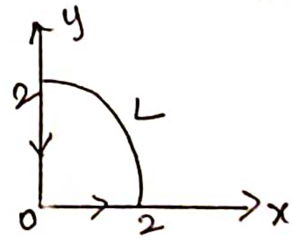
$$= \frac{\partial}{\partial x}(2xy) + \frac{\partial}{\partial y}y^2 + \frac{\partial}{\partial z}4yz = 2y+2y+4y = 8y$$

$$\iiint \nabla.\bar{F}\, dv = \int_0^1\int_0^1\int_0^1 8y\, dx\, dy\, dz$$

$$= \int_0^1\int_0^1 8yz\Big]_0^1 dx\, dy = \int_0^1\int_0^1 8y\, dx\, dy$$

$$= \int_0^1 \frac{8y^2}{2}\Big]_0^1 dx = 4\int_0^1 dx = 4\, x\Big]_0^1 = 4$$ Ans

Ex.2: Given $\vec{A} = r\sin\phi\, \hat{a}_r + r^2 \bar{a}_\phi$ in cylindrical co-ordinates. verify stokes theorem for the contour shown in fig.



$$\hat{A}.dl = (r\sin\phi\, \hat{a}_r + r^2 a_\phi).(dr\,\hat{a}_r + r\,d\phi\,\hat{a}_\phi + dz\,\hat{a}_z)$$

$$= r\sin\phi\, dr + r^3 d\phi$$

$$\therefore \oint_L \bar{A}.dl = \left[\int_{r=0}^{2} r\sin\phi\, dr\right]_{\phi=0} + \left[\int_{\phi=0}^{\pi/2} r^3 d\phi\right]_{r=2} + \left[\int_{r=2}^{0} r\sin\phi\, dr\right]_{\phi=\pi/2} = 0+4\pi-2 = (4\pi-2)$$

$$\nabla\times\bar{A} = \begin{vmatrix} \frac{1}{r}\hat{a}_r & \hat{a}_\phi & \frac{1}{r}\hat{a}_z \\ \partial/\partial r & \partial/\partial\phi & \partial/\partial z \\ r\sin\phi & r^3 & 0 \end{vmatrix} = (3r - \cos\phi)\,\hat{a}_z$$

$$\int_S(\nabla\times\bar{A}).ds = \int_{\phi=0}^{\pi/2}\int_{r=0}^{2}(3r-\cos\phi)\,r\,dr\,d\phi = \int_{\phi=0}^{\pi/2}(8-2\cos\phi)\,d\phi = 8\phi - 2\sin\phi\Big]_0^{\pi/2}$$
$$= (4\pi - 2)$$

$\therefore \int_S(\nabla\times\bar{A}).ds = \oint_L \bar{A}.dl$, $\therefore$ Stokes theorem is verified.

Null Identities:

The curl of the gradient of any scalar field is identically zero.

$$\nabla \times (\nabla V) = 0$$

If any vector is curl-free (irrotational vector field/conservative vector field), then it is expressed as gradient of scalar field.

If $\nabla \times E = 0$, then define scalar field V such that $E = -\nabla V$.

The divergence of curl of vector field is identically zero. $\nabla \cdot (\nabla \times A) = 0$

If a vector field is divergenceless (solenoidal), then it can be expressed as curl of another vector field.

If $\nabla \cdot B = 0$, define a vector field A such that $B = \nabla \times A$.

Helmholtz Theorem:

Any vector is a uniquely described within a region by its divergence and its curl. If divergence & curl of any vector $\bar{F}$ are given as, $\nabla \cdot \bar{F} = \rho_v$ & $\nabla \times \bar{F} = \rho_s$

where ρ_v - Source density of $\bar{F}$

ρ_s - Surface charge density of $\bar{F}$, both vanishing at ∞, then according to Helmholtz theorem, we can write a vector field $\bar{F}$ as sum of components $\bar{F_s}$ whose divergence is zero, $\nabla \cdot F_s = 0$ (solenoidal) & component $\bar{F_i}$ whose curl is zero. $\nabla \times \bar{F_i} = 0$ (irrotational)

$$\bar{F} = \bar{F_s} + \bar{F_i}$$

Also, divergence of curl of vector is zero & curl of grad of any scalar is zero.

Using two null identities, we can write $\bar{F_s}$ & $\bar{F_i}$ as follows.

$$\bar{F_s} = \nabla \times \bar{A}, \quad \bar{F_i} = \nabla \phi.$$

$\bar{A}$ & ϕ are vector and scalar quantities respectively.

Thus, any vector can be represented by Helmholtz theorem as, $\bar{F} = \nabla \phi + \nabla \times \bar{A}$.

It states that any vector satisfying equations with ρ_s & ρ_v vanishing at infinity can be written as sum of two vectors as irrotational (zero curl) and solenoid (zero divergence).

Problems:

1) Given two vectors $\bar{a} = 5\bar{i} + 3\bar{j}$, $\bar{b} = 3\bar{i} - 4\bar{j}$. Evaluate $\bar{a}+\bar{b}$, $\bar{a}-\bar{b}$, $\bar{b}-\bar{a}$, $\bar{a}\cdot\bar{b}$ & $\bar{a}\times\bar{b}$.

$\bar{a}+\bar{b} = 8\bar{i} - j$; $\bar{a}-\bar{b} = 2\bar{i} + 7\bar{j}$; $\bar{b}-\bar{a} = -2\bar{i} - 7j$;

$\bar{a}\cdot\bar{b} = 15 - 12 = 3$; $\bar{a}\times\bar{b} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 5 & 3 & 0 \\ 3 & -4 & 0 \end{vmatrix} = -29\bar{k}$.

2) Given two vector combination $\bar{A}+\bar{B} = 2\bar{i} + 3\bar{j} - 3\bar{k}$, $\bar{A}-\bar{B} = 4\bar{i} + \bar{j} + \bar{k}$. Find $\bar{A}$ & $\bar{B}$ in vector form.

$$\bar{A}+\bar{B} = 2\bar{i} + 3\bar{j} - 3\bar{k}$$
$$\bar{A}-\bar{B} = 4\bar{i} + \bar{j} + \bar{k}$$
$$2\bar{A} = 6\bar{i} + 4\bar{j} - 2\bar{k}$$
$$\therefore \bar{A} = 3\bar{i} + 2\bar{j} - \bar{k}$$
$$\bar{B} = -\bar{i} + \bar{j} - 2\bar{k}$$

3) Find the angle between two vectors $2\bar{i} - 2\bar{j} + \bar{k}$ & $2\bar{i} - \bar{j} - 2\bar{k}$.

$$\cos\theta = \frac{\bar{a}\cdot\bar{b}}{|\bar{a}|\,|\bar{b}|} = \frac{4+2-2}{\sqrt{4+4+1}\cdot\sqrt{9}} = \frac{4}{9}.$$

$$\theta = \cos^{-1}(4/9) = 63^\circ 36'.$$

4) If scalar potential is given by $\phi = xyz$, determine potential gradient & also prove $F = \text{grad}\,\phi$ is irrotational.

$$\phi = xyz$$

$$F = \nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z} = yz\bar{i} + xz\bar{j} + xy\bar{k}.$$

$$\nabla\times F = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yz & xz & xy \end{vmatrix} = \bar{i}(x-x) + \bar{j}(y-y) + \bar{k}(z-z) = 0.$$

$\therefore F = \text{grad}\,\phi$ is irrotational.

5) Find the curl of the vector field $\bar{H}$ at origin if

a) $\bar{H} = 2y\,\bar{i} + (z^2 - x^2)\,\bar{j} + 3y\,\bar{k}$

b) $\bar{H} = (2 - 1/r^2)\cos\phi\,\hat{u_r} + [2r - (2 + 1/r^2)\sin\phi\,\hat{u_\phi}$ [cylindrical]

c) $\bar{H} = R\sin\theta\,\hat{U_\phi}$ (Spherical coordinate).

a) $$\nabla \times H = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2y & z^2 - x^2 & 3y \end{vmatrix} = \bar{i}(3 - 2z) - \bar{j}(0) + \bar{k}(-2x - 2)$$

At origin, $x = y = 0$,

$\therefore \nabla \times H = 3\bar{i} - 2\bar{k}$

b) $$\nabla \times H = \frac{1}{r}\begin{vmatrix} \bar{i} & \bar{j}r & \bar{k} \\ \partial/\partial r & \partial/\partial \theta & \partial/\partial z \\ (2 - 1/r^2)\cos\phi & 2r^2 - (2 + 1/r^2)r\sin\phi & 0 \end{vmatrix}$$

$$= \frac{1}{r}\left[\bar{k}\,\partial/\partial r\left(2r^2 - (2 + 1/r^2)r\sin\phi\right)\right]$$

$$= \frac{1}{r}\left[\bar{k}\left(4r - (2 - 1/r^2)\sin\phi\right]\right. = \bar{k}\left[4 - \frac{1}{r}(2 - 1/r^2)\sin\phi\right]$$

At origin, $\nabla \times H = 4\bar{k}$.

c) $$\nabla \times H = \frac{1}{R^2\sin\theta}\begin{vmatrix} \bar{i} & \bar{j}R & \bar{k}R\sin\theta \\ \partial/\partial R & \partial/\partial \theta & \partial/\partial \phi \\ 0 & 0 & R^2\sin^2\theta \end{vmatrix}$$

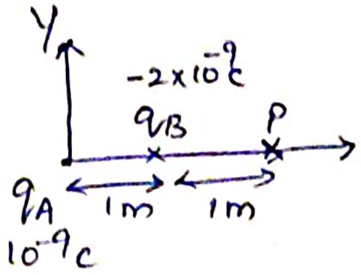
$$= \frac{1}{R^2\sin\theta}\left\{\bar{i}\,\partial/\partial\theta\left[R^2\left(\frac{1 - \cos 2\theta}{2}\right)\right] - \bar{j}R\,\partial/\partial R\,(R^2\sin^2\theta)\right\}$$

$$= \frac{1}{R^2\sin\theta}\left\{\bar{i}\,R^2\sin 2\theta - \bar{j}R(2R\sin^2\theta)\right\}$$

$$= \bar{i}\,2\cos\theta - \bar{j}\,2\sin\theta = 2\bar{i} \text{ (at origin)}$$

6) Calculate magnitude of field intensity on a unit charge at P on x-axis whose co-ordinates are $(x = 2m, y = 0)$ due to charges: A positive charge 10^{-9}C

is situated in air at origin (0,0) & a negative charge of -2×10^{-9} C is situated at $x=1m, y=0$



$$E_P = E_A + E_B$$

$$E_A = \frac{q}{4\pi\epsilon_0 r^2} = \frac{10^{-9}}{4\pi\epsilon_0\times2^2} = 2.25\ V/m$$

$$E_B = \frac{-2\times10^{-9}}{4\pi\epsilon_0\times1^2} = -18$$

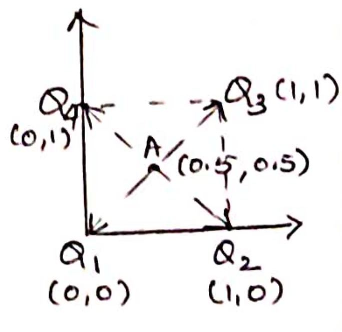
$$\therefore E_P = E_A + E_B = 2.25 - 18 = -15.75\ V/m.$$

7) Determine the potential difference between points a & b which are at distance of 0.5 m & 0.1 m, from a charge of -2×10^{-10} c.

$$V_{ba} = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{a}-\frac{1}{b}\right) = -(9\times10^9)\,2\times10^{-10}\left(\frac{1}{0.5}-\frac{1}{0.1}\right) = 14.4\ V$$

8) Four positive charges 10^{-9} c each are situated in xy plane at points (0,0) (0,1) (1,1) & (1,0) m. Find electric field & potential at (½, ½)

i) To find field intensity at A (½, ½)



Field due to Q_1 at A is $E_1 = \frac{Q_1}{4\pi\epsilon_0 (OA)^2}$

Field due to Q_3 at A is $E_3 = \frac{Q_3}{4\pi\epsilon_0 (AQ_3)^2}$

As charges are equal & $OA = AQ_3$; $|E_1| = |E_3|$ & they are oppositely directed.

$\therefore E_1 + E_3 = 0$, Similarly, $E_2 + E_4 = 0$.

Resultant field intensity is zero.

ii) To find potential at A.

All charges are equal & same distance of $r = \frac{\sqrt{2}}{2}$ m. from A. ∴ Resultant potential is four times that due to each.

Each charge produces potential at A $\Big\} = \frac{Q}{4\pi\epsilon_0 r} = \frac{9\times10^9\times10^{-9}\times2}{\sqrt{2}} = \frac{18}{\sqrt{2}}$ V

∴ Total potential $= \frac{18}{\sqrt{2}} \times 4 = 50.9 \approx 51$ V.

9) Three point charges in free space are located as follows: 5×10^{-8} C at (0,0) m, 4×10^{-8} C at (3,0) m 6×10^{-8} C at (0,4) m. i) Find potential, electric field intensity at (3,4) m. ii) What is total electric flux over a sphere of radius 5m with centre (0,0).

$$E_1 = \frac{Q_1}{4\pi\epsilon_0 r_1^2} = \frac{5\times10^{-8}}{4\pi\epsilon_0(3^2+4^2)} = 10.8\bar{i} + 14.4\bar{j} \text{ V/m}.$$

Similarly, $E_2 = 22.5\bar{j}$ V/m, $E_3 = -60\bar{i}$ V/m $\left(\because r_1 = 5\text{m},\ r_2 = 4\text{m},\ r_3 = 3\text{m}\right]$

$$\therefore E = E_1 + E_2 + E_3 = -49.2i + 36.9j$$
$$= 61.5 \angle 143.1^\circ.$$

Potential $V = \frac{Q_1}{4\pi\epsilon_0 r_1} + \frac{Q_2}{4\pi\epsilon_0 r_2} + \frac{Q_3}{4\pi\epsilon_0 r_3}$

$$= 9\times10^9\left[\frac{5\times10^{-8}}{5} + \frac{4\times10^{-8}}{4} + \frac{-6\times10^{-8}}{3}\right] = 0.$$

ii) Electric flux over Sphere of radius 5m.

Flux $= \phi_1 + \phi_2 + \phi_3 = (5+4-6)\,10^{-8} = 3\times10^{-8}$ C.

10) A point charge of 10 μc is located at (1,2,3) and another point charge of −3 μc is located at (3,0,2) in vaccum. Find the force between them.

$Q_1 = 10\mu c, \quad Q_2 = -3\mu c.$

$r_1 = \bar{a_x} + 2\bar{a_y} + 3\bar{a_z}, \quad r_2 = 3\bar{a_x} + 2\bar{a_z}$

$r = r_2 - r_1 = 2\bar{a_x} - 2\bar{a_y} - \bar{a_z}$

$|r| = \sqrt{4+4+1} = 3$

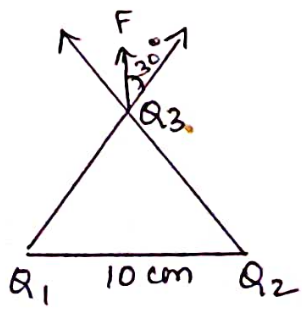
$\therefore$ Unit vector $\bar{a_r} = \dfrac{2\bar{a_x} - 2\bar{a_y} - \bar{a_z}}{3}$

Force $F = \dfrac{Q_1 Q_2}{4\pi\epsilon r^2}\hat{a_r}$

$= \dfrac{10\times10^{-6}\times(-3)\times10^{-6}}{4\pi\times8.854\times10^{-12}\times3^2} \cdot \left(\dfrac{2\bar{a_x} - 2\bar{a_y} - \bar{a_z}}{3}\right)$

$F = 4.44\times10^{-3}\,(2\bar{a_x} - 2\bar{a_y} - \bar{a_z})$ N.

11) Three concentrated charge of 0.25 μc are located at the vertices of equilateral triangle of 10 cm side. Find the magnitude and direction of force on one charge due to other two charges?



$Q_1 = Q_2 = Q_3 = 0.25\mu c$

Force on Q_3 due to Q_1

$|F_{13}| = \dfrac{Q_1 Q_3}{4\pi\epsilon r^2}$

$$|F_{13}| = \frac{0.25^{-2} \times 10^{-12}}{4\pi \times 8.854 \times 10^{-12} \times (0.1)^2}$$

$$= 0.05625 \text{ Nw}$$

force on Q_3 due to charge Q_2

$$|F_{23}| = \frac{Q_2 Q_3}{4\pi \epsilon r^2} = \frac{0.25^2 \times 10^{-12}}{4\pi \times 8.854 \times 10^{-12} \times (0.1)^2}$$

$$= 0.05625 \text{ Nw}$$

The resultant force

$$F = |F_{13}| \cos 30^\circ + |F_{23}| \cos 30^\circ.$$

$$= 2 \times 0.05625 \times \frac{\sqrt{3}}{2}$$

$$= 0.09743 \text{ Nw}.$$

12) Calculate electric field intensity at $P(3,-4,2)$ in free space by a) $Q_1 = 2\mu c$ at $(0,0,0)$

b) $Q_2 = 3\mu c$ at $(-1,2,3)$

c) $Q_1 = 2\mu c$ at $(0,0,0)$ and $Q_2 = 3\mu c$ at $(-1,2,3)$

a) Unit vector $\bar{a}_{21} = \dfrac{r_1 - r_2}{|r_1 - r_2|}$

$$= \frac{3\bar{a}_x - 4\bar{a}_y + 2\bar{a}_z}{\sqrt{9+16+4}}$$

$$= \frac{3\bar{a}_x - 4\bar{a}_y + 2\bar{a}_z}{\sqrt{29}}$$

Electric field intensity at P due to Q_1

$$E = \frac{Q_1}{4\pi\epsilon_0 r^2}\,\bar{a}_{21}$$

$$= \frac{2\times10^{-6}}{4\pi\times8.854\times10^{-12}(\sqrt{29})^2}\cdot\frac{3\bar{a}_x - 4\bar{a}_y + 2\bar{a}_z}{\sqrt{29}}$$

$$E = 345\,\bar{a}_x - 460\,\bar{a}_y + 230\,\bar{a}_z \ \ V/m.$$

b) $$\bar{a}_{31} = \frac{3-(-1)\,\bar{a}_x + \bar{a}_y\,(-4-2) + \bar{a}_z\,(2-3)}{\sqrt{16+36+1}}$$

$$= \frac{4\bar{a}_x - 6\bar{a}_y - \bar{a}_z}{\sqrt{53}}$$

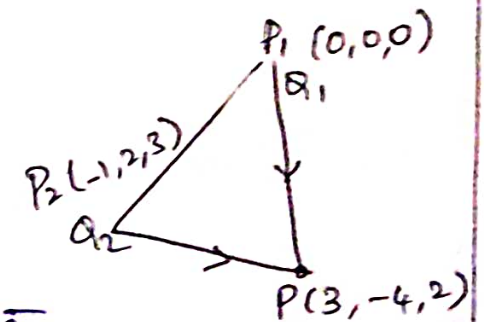
Electric field intensity at P due to Q_2

$$E = \frac{Q}{4\pi\epsilon r^2}\,\bar{a}_{31}$$

$$= \frac{3\times10^{-6}}{4\pi\times8.854\times10^{-12}\times(\sqrt{53})^2}\cdot\frac{4\bar{a}_x - 6\bar{a}_y - \bar{a}_z}{\sqrt{53}}$$

$$= 280\,\bar{a}_x - 420\,\bar{a}_y - 70\,\bar{a}_z \ \ V/m.$$

c) $$\bar{a}_{31} = \frac{3\bar{a}_x - 4\bar{a}_y + 2\bar{a}_z}{\sqrt{29}}$$



Electric field at P due to Q_1

$$E_1 = 345\,\bar{a}_x - 460\,\bar{a}_y + 230\,\bar{a}_z$$

Electric field at P due to Q_2

$$E_2 = 280\,\bar{a}_x - 420\,\bar{a}_y - 70\,\bar{a}_z$$

Total electric field intensity at P due to Q_1 & Q_2

$$E = E_1 + E_2$$

$$= 125\,\bar{a}_x - 880\,\bar{a}_y + 160\,\bar{a}_z \text{ V/m}.$$

13) Given the potential function $V = 4x + 3y$ in free space, find energy density.

$$V = 4x + 3y$$

$$E = -\nabla V = -\left(\frac{\partial}{\partial x}\bar{a}_x + \frac{\partial}{\partial y}\bar{a}_y + \frac{\partial}{\partial z}\bar{a}_z\right)(4x+3y)$$

$$= -4\,\bar{a}_x - 3\,\bar{a}_y$$

Magnitude of E is $\sqrt{16+9} = 5$ V/m.

$\therefore$ Energy density $= \frac{1}{2}\epsilon E^2$

$$= \frac{1}{2} \times 8.854 \times 10^{-12} \times 25$$

$$= 0.11 \times 10^{-9} \text{ J/m}^3.$$

UNIT II. ELECTROSTATICS

Coulomb's Law:

Force between any two point charges is
- directly proportional to product of magnitude of charges.
- inversely proportional to square of distance between them
- depends on medium in which two charges are placed.
- directed along the line joining two charges.

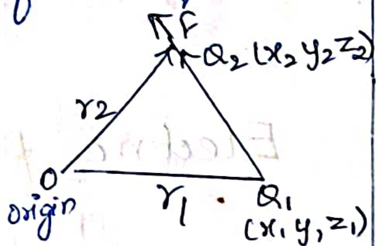
$$F = \frac{Q_1 Q_2}{4\pi \epsilon r^2} \text{ N/o}$$, where ϵ is permittivity of medium or dielectric constant.

Dielectric Constant: $\epsilon = \epsilon_0 \epsilon_r$ farad/m.

ϵ_r = Relative permittivity of medium.

$\epsilon_0 = 8.854 \times 10^{-12}$ F/m, permittivity of free space.

In vector form, $F = \frac{Q_1 Q_2}{4\pi \epsilon_0 r_{12}^2} \times \hat{a}_{12}$



where $\hat{a}_{12}$ - unit vector; $\hat{a}_{12} = \frac{r_2 - r_1}{|r_2 - r_1|}$

Principle of Superposition:

Consider a system consists of n point charges $Q_1, Q_2, \ldots Q_n$ at distances $r_1, r_2 \ldots r_n$ respectively. The force on n^{th} charge is given by vector sum of all individual forces. That is called Principle of Superposition.

Force F_i on i^{th} charge is given by,

$$F_i = \sum_{j=1}^{n} \frac{Q_i Q_j}{4\pi \epsilon_0 r_{ji}^2} \times a_{ij} \quad [i \neq j].$$

where Q_i is i^{th} charge, Q_j is any of charges other than Q_i

r_{ji} is distance between j^{th} & i^{th} charges.

Different charge Densities:

* Linear charge density: (Line charge density) $\rightarrow \lambda$

It is defined as total charge over a line.

$$\lambda = \lim_{\Delta l \to 0} \frac{\Delta Q}{\Delta l}$$

dimension of $\lambda \rightarrow$ charge per unit length.

* Surface charge density σ:

when charge is distributed over the surface, σ is defined as charge per unit area.

$$\sigma = \lim_{\Delta S \to 0} \frac{\Delta Q}{\Delta S} = \frac{dQ}{dS} \text{ coul/sq.m}$$

* Volume charge density ρ:

It is total charge per volume. It is given by

$$\rho = \lim_{\Delta V \to 0} = \frac{\Delta Q}{\Delta V} \text{ Coul/m}^3$$

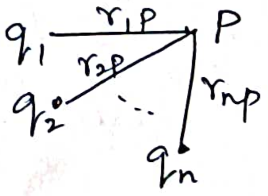
Electric field Intensity E:

It is defined as electric force per unit charge.

$$E = \frac{F}{q} = \frac{Q}{4\pi\epsilon r^2} \text{ New/Coul (or) V/m}$$

Field intensity due to discrete point charges:

Array of charges $q_1, q_2 \cdots q_n$



Field at point 'P' is evaluated by superposition principle

$$E_1 = \frac{q_1}{4\pi\epsilon r_{1P}^2}\hat{U}_{1P}, \quad E_2 = \frac{q_2}{4\pi\epsilon r_{2P}^2}\hat{U}_{2P} \cdots \quad E_n = \frac{q_n}{4\pi\epsilon r_{nP}^2}\hat{U}_{nP}.$$

Resultant field at P is $E = \frac{1}{4\pi\epsilon_0}\sum_{k=1}^{n}\frac{q_k}{r_{kP}^2}\hat{U}_{kP}$.

q_k - k^{th} charge, r_{kP} - Distance of k^{th} charge to P.

Electric field due to continuous charge:

Consider small elementry volume dv, volume charge density ρ, elemental charge ρdv, then, field intensity $E = \frac{1}{4\pi\epsilon_0} \iiint_v \frac{\rho dv}{r^2} \hat{u}_r$.

For surface charge distribution $E = \frac{1}{4\pi\epsilon_0} \iint_s \frac{\sigma ds}{r^2} \hat{u}_r$.

$\therefore$ Resultant field $E = \frac{1}{4\pi\epsilon_0} \left[\iiint_v \frac{\rho dv}{r^2} \hat{u}_r + \iint_s \frac{\sigma ds}{r^2} \hat{u}_r \right]$.

Problems:

1. Find the force of interaction between two charges spaced 10 cm apart in vaccum. Charges are 4×10^{-8}c & 6×10^{-5}c. If the same charges are seperated in kerosene ($\epsilon_r = 2$) by same distance, what is corresponding force of interaction?

i) $F = \frac{q_1 q_2}{4\pi\epsilon r^2} = \frac{4\times10^{-8}\times6\times10^{-5}}{4\pi\times8.854\times10^{-12}\times(0.1)^2} = 2.16$ Nw.

ii) $F = \frac{q_1 q_2}{4\pi\epsilon r^2} = \frac{2.16}{2} = 1.08$ Nw.

2. Force on a point charge situated 10 cm away from another point charge of same magnitude 0.1 N. Determine the magnitude of the charge.

$F = \frac{q_1 q_2}{4\pi\epsilon r^2} = \frac{q^2}{4\pi\times8.854\times10^{-12}\times(0.1)^2} = 0.1$

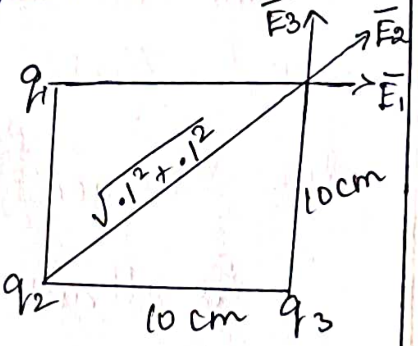
$\therefore q^2 = 1.112\times10^{-13}$; $q = 3.34\times10^{-7}$ c.

3. Two equal charge spheres repelled each other with force equal to weight of 109 mg. If centres 20 cm apart, Find charge on each.

$q_1 = q_2 = q$, $F = mg = 109\times10^{-3}\times9.8 = 1.0682$ Nw.

$1.0682 = \frac{q^2}{4\pi\epsilon_0\times(0.2)^2} \Rightarrow q^2 = 4.75\times10^{-12}$ c , $\therefore q = 2.179\ \mu$c.

4. Three equal positive charges 4 nc, each are located at three corners of square of side 10 cm, determine magnitude and direction of electric field at vacant corner point of square?



$$\bar{E} = \bar{E_1} + \bar{E_2} + \bar{E_3} \; ; \quad \epsilon_0 = 8.854 \times 10^{-12} \, F/m$$

$$\bar{E_1} = \frac{4\times10^{-9}}{4\pi\epsilon_0(0.1)^2} \angle 0^\circ = 3596.7\,\bar{i}$$

$$\bar{E_2} = \frac{4\times10^{-9}}{4\pi\epsilon_0(0.02)} \angle 45^\circ = 1271.6\,\bar{i} + 1271.6\,\bar{j}$$

$$\bar{E_3} = \frac{4\times10^{-9}}{4\pi\epsilon_0(0.1)^2} \angle 90^\circ = 3596.7\,\bar{j}$$

$\therefore$ Total field $\bar{E} = 6884.8 \angle 45^\circ$.

Electric Flux (Electric Displacement ψ):

Electric flux is equal to charge itself.

$$\boxed{\psi = Q \text{ coul}}$$

Electric flux density D:

It is electric flux per unit area.

$D = \frac{Q}{A}$ Coul/m^2 ; For sphere $D = \frac{Q}{4\pi r^2}$, but $E = \frac{Q}{4\pi\epsilon r^2}$

$\therefore \boxed{D = \epsilon E}$

Gauss Law:

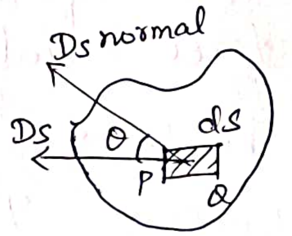
Diff. ampr, Wilson dw flux

The electric flux passing through any closed surface is equal to the total charge enclosed by the surface.

Consider a small area ds in a plane surface having charge Q.

P be a point in a element.

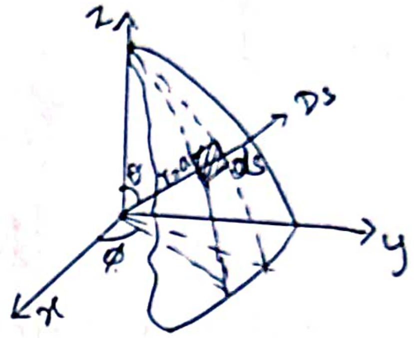
At each point, flux density D has value Ds.



Flux passing thro' closed surface is given by

$$\chi = \int d\chi = \int_s D_s \cdot ds = Q \quad \text{——} ①$$

Consider a charge Q at origin of spherical co-ordinate system, whose co-ordinates are r, θ, ϕ.



Electric field intensity $E = \dfrac{Q}{4\pi\epsilon r^2}$

Electric flux density $D = \dfrac{Q}{4\pi r^2}$

Consider small element of area ds on surface of sphere at distance r.

$$ds = r\,d\theta \; r\sin\theta\, d\phi = r^2 \sin\theta\, d\theta\, d\phi$$

Electric flux $\chi = \oint_s D\, ds = \int_s \dfrac{Q}{4\pi r^2} r^2 \sin\theta\, d\theta\, d\phi$

$$\chi = \frac{Q}{4\pi}\int_0^{2\pi}\int_0^{\pi} \sin\theta\, d\theta\, d\phi = \frac{Q}{4\pi}\int_0^{2\pi} [-\cos\theta]_0^{\pi}\, d\phi$$

$$= \frac{Q}{4\pi}\int_0^{2\pi} 2 \cdot d\phi = \frac{Q}{2\pi}\, \phi\Big]_0^{2\pi} = Q \quad \text{——} ②$$

∴ From ① & ②, $\boxed{\oint_s D \cdot ds = Q}$ Gauss Law. Proved.

In terms of volume charge density $\oint_s D \cdot ds = \int_v \rho\, dv$

Apply divergence theorem, $\oint_s D \cdot ds = \int_v \rho\, dv = \int_v \nabla \cdot D\, dv$

∴ $\boxed{\nabla \cdot \bar{D} = \rho}$

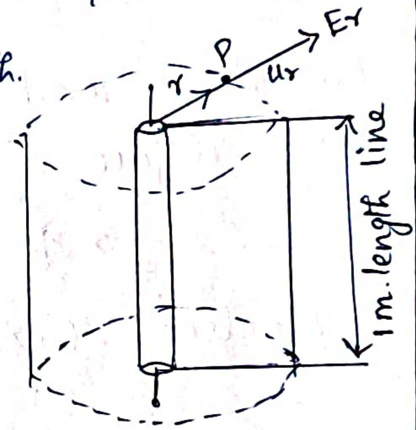
This is differential form or Point form of Gauss Law.

Applications of Gauss Law :

- Applied for evaluation of electric field intensity, if it is possible to choose the closed surface such that electric field has normal component which is a single valued or zero at every point on surface.

Ex.1: Determine the field at a distance r from an infinite line charge of length λ coul/m.

Infinite charged line with 1m. length.



Imagine coaxial cylindrical surface surrounds the charged line over a mtr length. This is called 'Gaussian surface'.

Electric field at any point is radial & independent of both positions via. along & angular position around the wire.

Electric field is uniform & directed outward radially.

Apply Gauss law for a charged line,

$$\iint_S E.\hat{n}\, ds = 2\pi r E_r = \frac{\lambda}{\epsilon_0} \qquad \therefore E_r = \frac{\lambda}{2\pi\epsilon_0 r}\hat{u}_r .$$

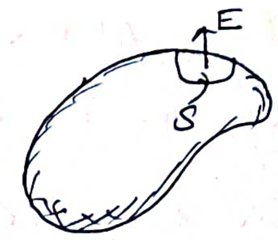
What is the field inside the charged conductor?

Net charge of the charged conductor resides on its surface. Net charge enclosed by each of Gaussian surface is zero.

ex-2: Consider a closed pill box shaped surface S resting on the surface of a charged conductor. Find an expression for the field just outside conductor?

Let, charge on the conductor is, surface charge density function σ.

Element surface area Δs.



Apply Gauss law to pillbox S,

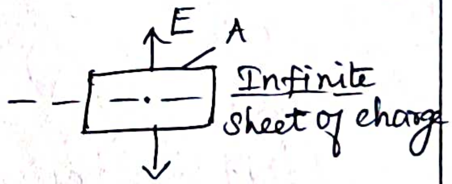
Surface integral of Electric field $E = E_1 \hat{n}\, ds$

This is charge element corresponds to area Δs divided by ϵ_0.

$$\therefore \quad \frac{\Delta q}{\epsilon_0} = \frac{\sigma \Delta s}{\epsilon_0}. \qquad E \cdot \hat{n}\, \Delta s = \frac{\sigma \Delta s}{\epsilon_0}$$

$$\therefore E = \frac{\sigma}{\epsilon_0}\hat{n}.$$

ex.3: Find electric field intensity due to uniformly charged infinite plane sheet with surface charge density of σ coul/m^2.

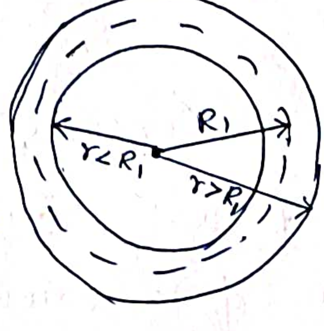


Field is perpendicular to the surface of infinite plane sheet of charge.

$$\iint E \cdot \hat{n}\, ds = 2EA = \frac{\sigma A}{\epsilon_0}; \qquad \therefore E = \frac{\sigma}{2\epsilon_0}.$$

ex.4: Determine the variation of field from point to point due to i) Single spherical cell of charge with radius R_1
ii) two concentric spherical shells of radii R_1 & R_2.
iii) Spherical volume distribution of charge of radius R & density ρ.

i) Total charge Q is distributed over an shell of radius R_1 in free space.



At $r < R_1$, no charge enclosed by surface.

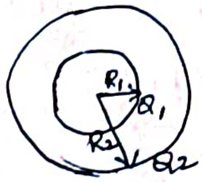
$$\therefore \oiint_S D \cdot \hat{n}\, ds = \epsilon_0 \oiint_S E \cdot \hat{n}\, ds = 0.$$

At $r \geq R_1$, $\bar{D}$ over spherical surface is charge itself.

$$\epsilon_0 \oiint E \cdot \hat{n}\, ds = \epsilon_0 E(4\pi R_1^2) = Q. \qquad \therefore E = \frac{Q}{4\pi\epsilon_0 R_1^2}$$

ii) At $r < R_1$, $E = 0$.

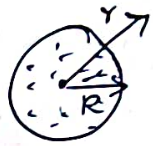
At $R_1 \le r \le R_2$, $E = \dfrac{Q_1}{4\pi\epsilon_0 r^2}\hat{U}_r$



Just outside shells, charges Q_1 & Q_2 are present.

$\therefore$ At $r \ge R_2$, $E = \dfrac{Q_1 + Q_2}{4\pi\epsilon_0 r^2}\hat{u}_r$.

iii) Charge contained by sphere of radius $r < R$ is proportional to volume which is proportional to cube of radius, so charge Q_r at any radius $r < R$ is related to volume charge Q_t as follows:



$$\frac{Q_r}{Q_t} = \left(\frac{r}{R}\right)^3 ; \quad Q_r = Q_t\left(\frac{r}{R}\right)^3.$$

Field intensity at $r < R$, $E = \dfrac{Q_r}{4\pi\epsilon_0 r^2}\hat{U}_r = \dfrac{Q_t\, r}{4\pi\epsilon_0 R^3}\hat{U}_r$

At $r \ge R$, $E = \dfrac{Q_t}{4\pi\epsilon_0 r^2}\hat{u}_r$..

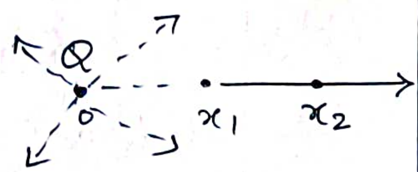
Electric Potential:

Absolute potential at a point is defined as the work done in moving a unit positive charge from infinity to a given point in electric field.

$$\boxed{V = \frac{Q}{4\pi\epsilon r} \text{ volts}}$$

Potential is a scalar quantity.

Consider uniform electric field E is point charge q. If that test charge moved from x_2 to x_1 opposite to field orientation, there will be work done against force.



Work = Force x distance $= |E| q d$

Work done / charge $= E.d$.

Potential Difference: → Work done in moving a test charge from one point to another in electric field.

Work done per charge from x_2 to x_1 is

$$V_{21} = \int_{x_2}^{x_1} dV = -\int_{x_2}^{x_1} E\,dx = -\int_{x_2}^{x_1} \frac{Q}{4\pi\epsilon_0}\frac{dx}{x^2}$$

$$= -\frac{Q}{4\pi\epsilon_0}\left[-\frac{1}{x}\right]_{x_2}^{x_1} = \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{x_1} - \frac{1}{x_2}\right]$$

$$V_{21} = V_1 - V_2 = \frac{Q}{4\pi\epsilon_0 x_1} - \frac{Q}{4\pi\epsilon_0 x_2} .$$

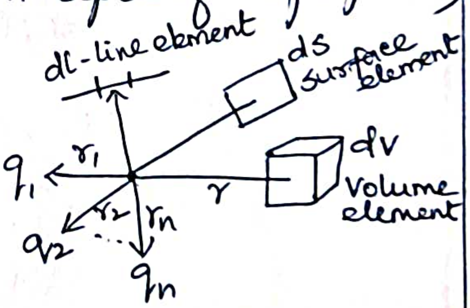
$\therefore V_1 = \frac{Q}{4\pi\epsilon_0 x_1}$

$V_2 = \frac{Q}{4\pi\epsilon_0 x_2}$.

In general, $V = \frac{Q}{4\pi\epsilon_0 r}$.

Potential due to many charges (Principle of Superposition)

consider charges $q_1, q_2 \cdots q_n$



Linear charge density λ coul/m.

Surface charge density σ coul/m².

Volume charge density ρ coul/m³.

Potential due to charges $V_p = \frac{1}{4\pi\epsilon_0}\left(\frac{q_1}{r_1} + \frac{q_2}{r_2} + \cdots + \frac{q_n}{r_n}\right)$

Potential due to line charge $V_L = \frac{1}{4\pi\epsilon_0}\int \frac{\lambda(r)}{r}\,dl$

Potential due to surface $V_S = \frac{1}{4\pi\epsilon_0}\iint \frac{\sigma(r)}{r}\,ds$.

Potential due to volume $V_{vol} = \frac{1}{4\pi\epsilon_0}\iiint \frac{\rho(r)}{r}\,dv$.

$\therefore$ Total $V = V_p + V_L + V_s + V_{vol} = \frac{1}{4\pi\epsilon_0}\left[\sum_{k=1}^{n}\frac{q_k}{r_k} + \int \frac{\lambda(r)}{r}\,dl + \iint \frac{\sigma(r)}{r}\,ds + \iiint \frac{\rho(r)}{r}\,dv\right]$

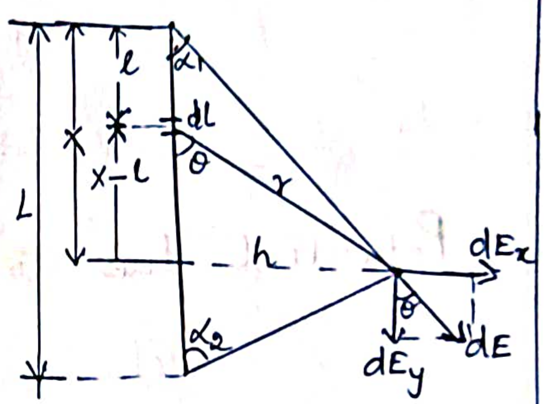
Relation between Potential & Electric field intensity:

Potential gradient is electric field intensity

$$\boxed{\vec{E} = -\nabla V}$$

Electric field intensity and Potential due to uniformly charged line :

Consider a uniformly charged line of length L whose linear charge density ρ_l coul/m. Consider a small element dl at a distance 'l' from one end of charged line. Let 'P' be any point at a distance 'r' from the element dl.



Electric field at a point P due to charge element $\rho\,dl$ is

$$dE = \frac{\rho_l\,dl}{4\pi\epsilon r^2}$$

x & y components of electric field are

$$dE_x = dE\sin\theta \quad , \quad dE_y = dE\cos\theta$$

Then, $$dE_x = \frac{\rho_l\,dl\sin\theta}{4\pi\epsilon r^2}.$$

From fig. $x - l = h\cot\theta$ & $\frac{h}{r} = \sin\theta$

$-dl = -h\,\mathrm{cosec}^2\theta\,d\theta$. $\quad r = h\,\mathrm{cosec}\,\theta$.

$$\therefore dE_x = \frac{\rho_l\,\cancel{dl}\sin\theta\,d\theta}{4\pi\epsilon h}.$$

Electric field Ex due to entire length of line charge is

$$E_x = \int_{\alpha_1}^{\pi-\alpha_2} \frac{\rho_l\sin\theta\,d\theta}{4\pi\epsilon h} = \frac{\rho_l}{4\pi\epsilon h}\left[-\cos\theta\right]_{\alpha_1}^{\pi-\alpha_2} = \frac{\rho_l}{4\pi\epsilon h}\left[\cos\alpha_1 + \cos\alpha_2\right]$$

For y component of E, $dE_y = \frac{\rho_l}{4\pi\epsilon h}\cos\theta\,d\theta$.

$$E_y = \int_{\alpha_1}^{\pi-\alpha_2} \frac{\rho_l}{4\pi\epsilon h}\cos\theta\,d\theta = \frac{\rho_l}{4\pi\epsilon h}\left[\sin\theta\right]_{\alpha_1}^{\pi-\alpha_2} = \frac{\rho_l}{4\pi\epsilon h}\left[\sin\alpha_2 - \sin\alpha_1\right]$$

Case 1: If point P is at bisector of a line, then $\alpha_1 = \alpha_2 = \alpha$ $\therefore E_y = 0$, E becomes E_x.

$$E = \frac{\rho_L}{2\pi\epsilon h} \cos\alpha.$$

Case 2: If line is infinitely long then $\alpha = 0$

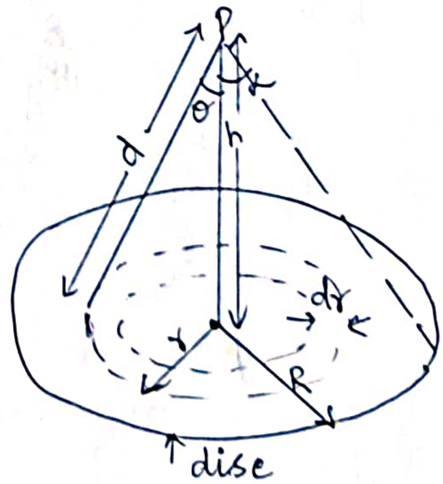
$E_y = 0$, E becomes E_x, $\therefore E = \dfrac{\rho_L}{2\pi\epsilon h}$.

Potential difference $V_{ba} = -\int_b^a E\,dr = -\int_b^a \dfrac{\rho\,dr}{2\pi\epsilon r} = \dfrac{\rho}{2\pi\epsilon} \ln\dfrac{b}{a}$

[Take $h = r$]

Electric field intensity & Potential at any point due to charged circular disc:

Consider a circular disc of radius 'R' is charged uniformly with charge density ρ_s Coul/m². Let P be any point on the axis of disc at a distance 'h' from the centre. Consider ring of radius r and thickness dr. Area of ring $ds = 2\pi r\,dr$.



Field intensity at Point P $\Big\}$ $dE = \dfrac{\rho_s\,ds}{4\pi\epsilon d^2}$

Since horizontal component of electric field intensity is zero, Vertical component is given by

$$dE_y = \frac{\rho_s\,ds\cos\theta}{4\pi\epsilon d^2} = \frac{\rho_s\,2\pi r\,dr}{4\pi\epsilon d^2}\cos\theta.$$

From fig. $r = h\tan\theta$, $dr = h\sec^2\theta\,d\theta$, $d = \dfrac{r}{\sin\theta}$.

then $dE_y = \dfrac{\rho_s\,2\pi r h \sec^2\theta\,d\theta\,\cos\theta\,\sin^2\theta}{4\pi\epsilon r^2}$

$$dE_y = \frac{\rho_s h \sec\theta \cdot \sin^2\theta \, d\theta}{2\epsilon r} = \frac{\rho_s \sec\theta \cdot \sin^2\theta \, d\theta}{2\epsilon \tan\theta} \quad \left[\because \frac{r}{h} = \tan\theta\right]$$

$$= \frac{\rho_s}{2\epsilon} \sin\theta \, d\theta . \quad [\because \sec\theta \sin\theta = \tan\theta]$$

Total electric field due to charged disc

$$E = \int_{\theta=0}^{\alpha} dE_y = \frac{\rho_s}{2\epsilon} \int_0^{\alpha} \sin\theta \, d\theta = \frac{\rho_s}{2\epsilon} [1 - \cos\alpha] = \frac{\rho_s}{2\epsilon} \left[1 - \frac{h}{\sqrt{h^2+R^2}}\right]$$

Electric Potential $V = -\int_d^0 E . dx$.

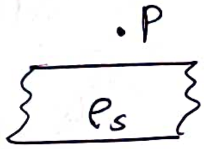
$$V = \int_d^0 -\frac{\rho_s}{2\epsilon}(1-\cos\alpha)\, dx = \frac{\rho_s}{2\epsilon}(1-\cos\alpha) . d$$

$$V = \frac{\rho_s}{2\epsilon}\left[1 - \frac{h}{\sqrt{h^2+R^2}}\right) \sqrt{h^2+R^2} = \frac{\rho_s}{2\epsilon}\left(\sqrt{h^2+R^2} - h\right) \text{ volts}$$

Electric field intensity and potential at any point due to infinite plane sheet of charge:

Consider an infinite plate sheet which is uniformly charged with charge density ρ_s coul/m².

Field intensity $E = \frac{\rho_s}{2\epsilon}(1-\cos\alpha) = \frac{\rho_s}{2\epsilon}$ $[\alpha = 90^\circ]$



Potential $V = -\int_d^0 E\, dx = \frac{\rho_s}{2\epsilon} d = \frac{\rho_s}{2\epsilon}\sqrt{h^2+R^2}$ volts.

Similarly, electric field at point P due to two oppositely charged infinite plane $\Big\} = \frac{\sigma}{2\epsilon} + \frac{\sigma}{2\epsilon} = \frac{\sigma}{\epsilon}$.

Single Shell of charge:

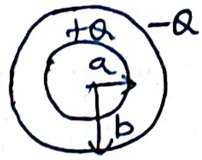
Let charge Q is uniformly distributed over a sphere of radius r.

By Gauss Law, at $r < a$, $E = 0$. [No charge enclosed by the surface]

At $r > a$, $E = \frac{Q}{4\pi\epsilon r^2}$, $\therefore V = \frac{Q}{4\pi\epsilon r}$ $\left[V = -\int E\,dr\right]$

Potential between two concentric shells:

Consider two shells of inner radius 'a', outer radius 'b'. Let Q be charge distributed over inner shell & −Q be distributed over outer shell.



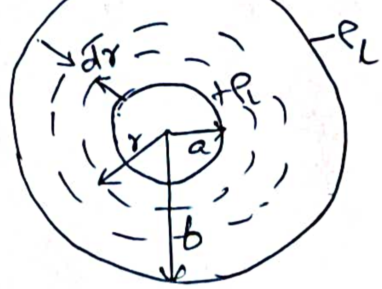
Potential difference $V = -\int_b^a E\,dr = \int_b^a \frac{Q}{4\pi\epsilon r^2}\,dr$ $\left[\because E = \frac{Q}{4\pi\epsilon r^2}\right]$

$$V = \frac{Q}{4\pi\epsilon}\left[\frac{1}{r}\right]_b^a = \frac{Q}{4\pi\epsilon}\left[\frac{1}{a} - \frac{1}{b}\right]$$

Co-axial Cylinders:

Consider two co-axial cylinders of inner radius 'a' & outer radius 'b', distributed with charge density ρ_l coul/m. Inner cylinder has charge density ρ_l and outer has $-\rho_L$. Let E be electric field at dist. r from axis of cylinders of length l. Element area $ds = 2\pi r l$

By Gauss law, $\oint D.ds = \rho_l l$



$$\epsilon \int E.ds = \rho_l l$$

$$E.2\pi r l = \frac{\rho_l l}{\epsilon}$$

$$E = \frac{\rho_l}{2\pi\epsilon r}$$

Potential difference $V = -\frac{\rho_l}{2\pi\epsilon}\int_b^a \frac{dr}{r} = -\frac{\rho_L}{2\pi\epsilon} \ln a/b$

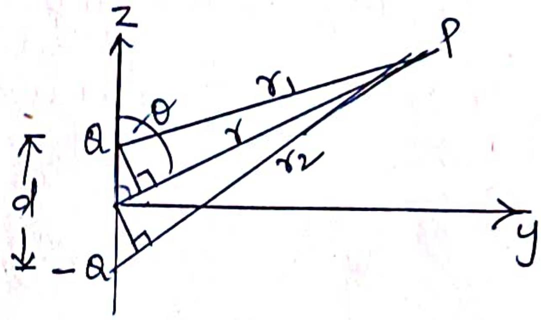
$$V = \frac{\rho_l}{2\pi\epsilon} \ln \frac{b}{a}.$$

Electric Dipole:

An electric dipole is two equal and opposite charges seperated by a small distance. The product of charge & spacing is electric dipole moment.

Let Q and −Q be the two charges seperated by a small distance d.

Dipole moment $m = Qd$.



Let P be any point at distance of r_1, r_2 & r from +Q, −Q and mid point of pole.

Potential at P due to +Q is $V_1 = \frac{Q}{4\pi\epsilon r_1}$

Potential at P due to −Q is $V_2 = \frac{-Q}{4\pi\epsilon r_2}$.

Resultant potential $V = V_1 + V_2 = \frac{Q}{4\pi\epsilon}\left[\frac{1}{r_1} - \frac{1}{r_2}\right]$.

If point P is too far away from dipole, then

$r_1 = r - \frac{d}{2}\cos\theta$, $r_2 = r + \frac{d}{2}\cos\theta$. $\frac{r - r_1}{d/2} = \cos\theta$ $\frac{r_2 - r}{d/2} = \cos\theta$

Potential at P due to dipole $\Big\}$ $V = \frac{Q}{4\pi\epsilon}\left[\frac{1}{r - \frac{d}{2}\cos\theta} - \frac{1}{r + \frac{d}{2}\cos\theta}\right]$

$$= \frac{Q}{4\pi\epsilon}\left[\frac{d\cos\theta}{(r - \frac{d}{2}\cos\theta)(r + \frac{d}{2}\cos\theta)}\right]$$

$$= \frac{Q}{4\pi\epsilon}\left[\frac{d\cos\theta}{r^2 - (\frac{d}{2}\cos\theta)^2}\right] = \frac{Q}{4\pi\epsilon}\left(\frac{d\cos\theta}{r^2}\right)$$

$$V = \frac{m\cos\theta}{4\pi\epsilon r^2} \quad \left[\because \left(\frac{d}{2}\right) << r^2,\ m = Qd\right]$$

From this, potential is directly proportional to dipole moment & inversely proportional to square of the distance.

Conductors in static Electric Field:

Conductor has large quantity of charge that is free to move.

In an isolated conductor, when an external field is applied, positive charges are pushed along same direction as applied field, while negative charges move in opposite direction.

The free charges do two things:

i) They accumulate on the surface of conductor and form an induced surface charge.

ii) The induced charges set up an internal induced field E_i, which cancels external applied field.

* A perfect conductor cannot contain an electro static field within it.

A conductor is called an equipotential body (ie) potential is same everywhere in conductor that is based on $E = -\nabla V = 0$.

* Based on Ohm's law, $J = \sigma E$

$E \to 0$ because $\sigma \to \infty$ in perfect conductor

* Based on Gauss law,

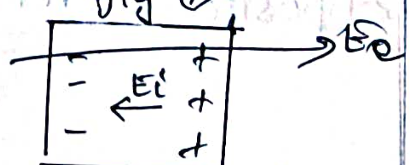
if $E = 0$, $\rho_v = 0$.

Under static conditions,

$E = 0$, $\rho_v = 0$, $V_{ab} = 0$

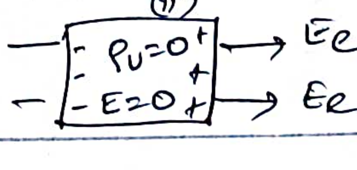
inside a conductor.

fig ①



Conductor under
i) applied field
ii) static condition.

(ii)



Dielectric in Static Electric Field:

* The conduction properties depend on energies in valence & conduction bands.

* If forbidden energy gap between valence and conduction band of a material is high, it requires large applied energy to conduct; This material is called dielectric.

* In dielectric, electrons are tightly bounded.

Types: 1) Non-polar molecules. 2) Polar molecules

1) The molecules whose centre of positive and negative charges coincide are called non-polar molecules.

2) The molecules whose centre of charges are displaced from each other are called polar molecules.

When dielectric material madeup of non polar molecules placed in field, displacement of negative and positive charges takes place in opposite directions and produce a dipole which is aligned with electric field.

UNIT II: CONDUCTORS AND DIELECTRICS.

Conductors & Dielectrics:

The electrical conduction properties of different elements and compounds can be explained in terms of electrons having energies in valance and conduction bands. A material maybe classified as

* Conductor * dielectric * Semiconductor

- conductor: If valence band merges smoothly into conduction band, then additional kinetic energy maybe given to valence electrons by an external field resulting in a electron flow. The solid is called metallic conductor.
  - There is no forbidden energy gap for metal, the required applied energy is small.
  - It has excellent conductivity.

- Dielectric: If forbidden energy gap between valence and conduction band of a material is high, it requires large applied energy to conduct. That material is called dielectric.
  - here, electrons are tightly bounded.

- Semiconductor: Materials lie between conductor and dielectric are called as semiconductor.

In electromagnetics, conductors and dielectrics are defined on the ratio of conduction to displacement current.

Ratio of conduction to displacement current is $\frac{\sigma}{\omega\epsilon}$.

σ - Conductivity of material. ω - angular frequency.
ϵ - permittivity of medium.

For good conductors, $\frac{\sigma}{\omega\epsilon} >> 1$.

For good dielectric, $\sigma/\omega\epsilon < 1$.

Molecules of dielectric material may be two types: * Non-polar molecules * Polar molecules.

Non-polar molecules: The molecules whose centre of positive and negative charges coincide are called non-polar molecules.

Polar molecules: The molecules whose centre of charges are displaced from each other are called Polar molecules.

Current and Current Density:

Electric charges in motion constitute a current. Current is rate of movement of charge passing a given reference point. Unit → ampere

$$I = dQ/dt.$$

The increment of charge current ΔI crossing an incremental surface ΔS normal to current density is $\Delta I = J \cdot \Delta S$

$$\therefore I = \int_S J \cdot ds$$ where J - current density.

Boundary Conditions:

The conditions pertaining to normal component of the displacement, tangential component of electric field intensity and refraction lines of forces are called boundary conditions. The statements of boundary conditions are

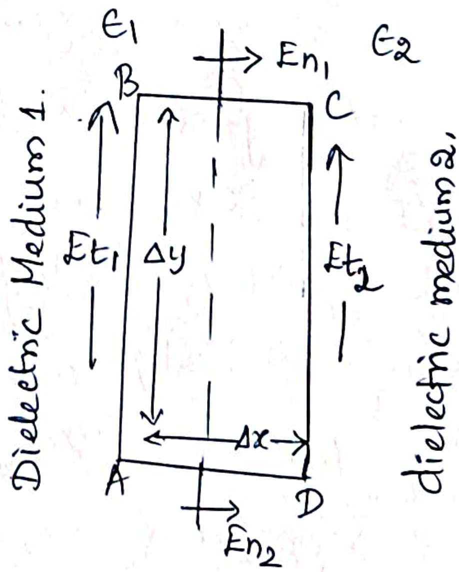
i) The tangential component of electric field E is continuous at the surface. (ie) E is same just outside the surface as it is just inside the surface.

ii) The normal component of electric flux density is continuous if there is no surface charge density. Otherwise D is discontinuous by an amount equal to surface charge density.

Between dielectrics:

a) Consider a interface between two dielectrics of dielectric constants ϵ_1 & ϵ_2 in electric field.

Fig. shows boundary surface between two dielectric media.



Consider a rectangle of length Δy & width Δx. In electric field, voltage around any closed path must be zero.

$$V = \oint E.dl = 0.$$

Apply this to rectangular path ABCD, in which AB is inside medium 1 and CD is inside medium 2.

$$\oint E.dl = E_{t_1}\Delta y + E_{n_1}\Delta x - E_{t_2}\Delta y - E_{n_2}\Delta x.$$

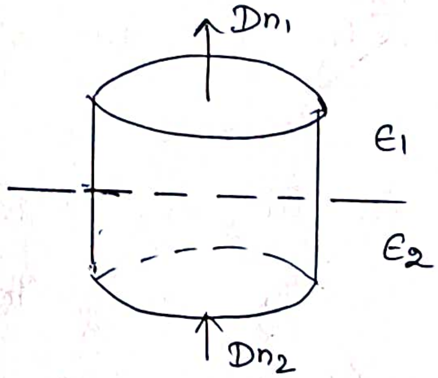
Where, E_{t_1} & E_{t_2} are average tangential components of E along paths AB & CD and E_{n_1} & E_{n_2} are average normal components of E along the paths BC & AD. As the sides AB & CD are brought closer together, lengths BC & AD approach zero. (ie) $\Delta x \to 0$.

$$\therefore E_{t_1}\Delta y - E_{t_2}\Delta y = \oint E.dl = 0.$$

$\boxed{E_{t_1} = E_{t_2}}$ The tangential component of E is continuous at the boundary.

b) Consider pill box at the boundary of two dielectric constants ϵ_1 & ϵ_2, it is assumed to be no free charges on the boundary surface.

Apply Gauss law, since no charges enclosed by the surface, the surface integral of electric flux density over the pillbox surface is zero.



$$\int_s D_{n_1} ds - D_{n_2} ds = 0$$ where D_{n_1} & D_{n_2} are electric flux density in medium 1 & 2
ds - Surface area of pill box.

$\boxed{D_{n_1} = D_{n_2}}$

The normal components of electric flux densities are continuous across boundary (ie) they are equal.

If charges are enclosed by pill box, then,

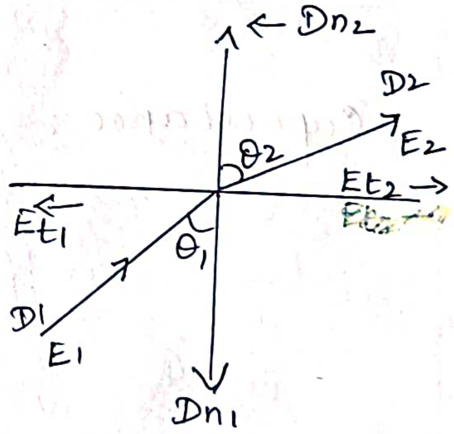
$$\int_s D_{n1}\, ds - \int_s D_{n2}\, ds = \int \rho_v\, dv = \int \rho_v\, \Delta x\, ds.$$

where ρ_v - volume charge density, Δx - height of pillbox.

If $\Delta x \to 0$, $\lim_{\Delta x \to 0} \rho_v \Delta x = \rho_s$ (surface charge density)

Then, $\boxed{D_{n1} - D_{n2} = \rho_s}$ Normal component of flux density is discontinuous by amount of surface charge density.

c) Consider two dielectric media 1 & 2 seperated by a charge free boundary.



Electric field intensity E_1 is incident in medium 1 at angle of θ_1 & E_2 is refracted in medium 2 with angle of θ_2.

D_1 & D_2 - electric displacement in medium 1 & 2 respectively.

Normal Components of D are

$$D_{n1} = D_1 \cos\theta_1 \quad , \quad D_{n2} = D_2 \cos\theta_2$$

Tangential components of E are

$$E_{t1} = E_1 \sin\theta_1 \quad , \quad E_{t2} = E_2 \sin\theta_2 .$$

Apply boundary conditions

$$D_{n1} = D_{n2} \quad , \quad E_{t1} = E_{t2} .$$

$\therefore D_1 \cos\theta_1 = D_2 \cos\theta_2$; $E_1 \sin\theta_1 = E_2 \sin\theta_2$.

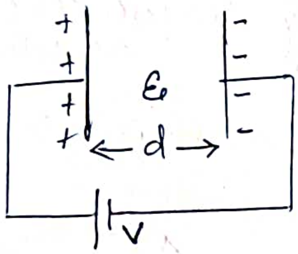
$$\frac{E_1}{D_1}\tan\theta_1 = \frac{E_2}{D_2}\tan\theta_2.$$

But $D_1 = \epsilon_1 E_1$ & $D_2 = \epsilon_2 E_2$,

then $$\frac{E_1}{\epsilon_1 E_1}\tan\theta_1 = \frac{E_2}{\epsilon_2 E_2}\tan\theta_2.$$

$$\boxed{\frac{\tan\theta_1}{\tan\theta_2} = \frac{\epsilon_1}{\epsilon_2}}$$

Capacitor: It is an electronic device composed of two conductors seperated by dielectric medium, and which store equal & opposite charges. ($\pm Q$).



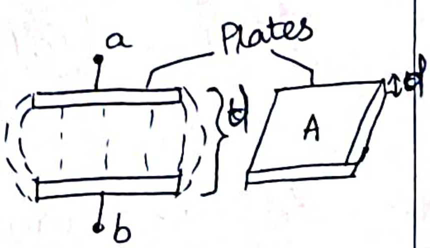
capacitance: It is defined as ratio of magnitude of charge on either of the conductor to the potential difference between the conductors.

$$C = \frac{Q}{V} \text{ Farad (or) Coul/Volt.}$$

Parallel Plate Capacitor:

It consists of pair of parallel plates with surface area 'A' seperated by distance 'd' through dielectric permittivity $\epsilon = \epsilon_0 \epsilon_r$.

Capacitor is charged by connecting a & b to a source of potential difference.



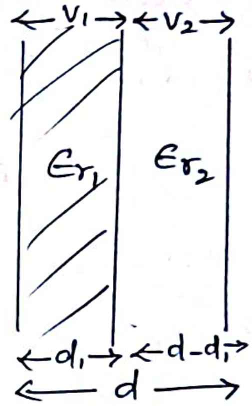
Assume uniform charge density over plate surface σ C/m² and also across dielectric.

$$D = \frac{Q}{A} \quad ; \quad \therefore Q = DA = \epsilon E A.$$

But $E = V/d$, then $Q = \frac{\epsilon A \cdot V}{d}$

So, capacitance $C = \frac{Q}{V} = \frac{\epsilon A}{d} = \frac{\epsilon_0 \epsilon_r A}{d}$ farad.

Capacitance of a parallel plate capacitor having two dielectric media:



Consider parallel plate capacitor consist of two dielectrics. Relative permittivity of dielectric medium 1 & 2 are ϵ_{r_1} and ϵ_{r_2} respectively. If potential across capacitor is V, potential difference across medium 1 & 2 are V_1 & V_2 respectively. $V = V_1 + V_2$

Let E_1 & E_2 be field intensities in that mediums

$$V_1 = E_1 d_1 \quad ; \quad V_2 = E_2 (d - d_1).$$

$$\therefore V = V_1 + V_2 = E_1 d_1 + E_2 (d - d_1).$$

Flux densities $D = Q/A$ will be same for both media.

$$\therefore E_1 = \frac{D}{\epsilon_0 \epsilon_{r1}} = \frac{Q}{A \epsilon_{r_1} \epsilon_0} \quad ; \quad E_2 = \frac{D}{\epsilon_0 \epsilon_{r_2}} = \frac{Q}{A \epsilon_{r_2} \epsilon_0}$$

Hence, $V = \frac{Q}{A \epsilon_0} \left[\frac{d_1}{\epsilon_{r_1}} + \frac{d - d_1}{\epsilon_{r2}} \right].$

$$Q/V = A \epsilon_0 \Big/ \left(\frac{d_1}{\epsilon_{r1}} + \frac{d - d_1}{\epsilon_{r2}} \right)$$

Capacitance $C = \frac{Q}{V} = \frac{A \epsilon_0 \epsilon_{r_1} \epsilon_{r_2}}{d_1 \epsilon_{r_2} + (d - d_1) \epsilon_{r_1}}$

Ex. 1: Determine the capacitance of parallel plate capacitor composed of tin foil sheets 25 cm² side seperated by a glass dielectric 0.5 cm thickness with relative permittivity 6.

Capacitance $C = \frac{\epsilon_0 \epsilon_r A}{d} = \frac{8.854 \times 10^{-12} \times 6 \times 0.25 \times 0.25}{0.005} = 663\ pf.$

Ex. 2: Two parallel conducting plates 3 cm apart in air are connected to potential difference of 72 kV. If mica sheet ($\epsilon_r = 4$) of thickness 1 cm is introduced between plates, determine field intensities in air & mica.

Field intensity in air gap $E = V/t = \frac{72}{3} = 24\ kV/cm.$

Thickness of air film reduced to . $t_1 = 2$ cm after inserting mica

Thickness of mica $t_2 = 1$ cm.

$E_1 = D/\epsilon_0$, $E_2 = \frac{D}{\epsilon_0 \epsilon_r} = \frac{D}{4\epsilon_0}$; $\therefore E_1 = 4E_2$

$E_1 t_1 + E_2 t_2 = V$

$4E_2 t_1 + E_2 t_2 = 72$

$4E_2 \times 2 + E_2 \times 1 = 72$

$\therefore E_2 = \frac{72}{9} = 8\ kV/cm.$, So, $E_1 = 4 \times 8 = 32\ kV/cm.$

Ex. 3: Determine the capacitance of a capacitor consists of two parallel plates 30 cm × 30 cm area, seperated by 5 mm in air.

Capacitance $C = \frac{\epsilon_0 \epsilon_r A}{d} = \frac{8.854 \times 10^{-12} \times 30 \times 30 \times 10^{-4}}{5 \times 10^{-3}}$

$C = 159.3 \times 10^{-12}$ farad.

Capacitance of an isolated Sphere:

Consider a sphere of radius r having Charge Q. The potential is work done per unit charge in carrying a positive charge from infinity to sphere.

Potential. $V = -\int_{\infty}^{r} E.dr$, but $E = \frac{Q}{4\pi\epsilon r^2}$

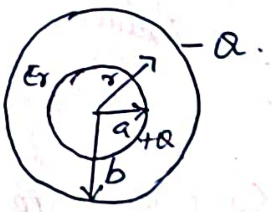
$$= -\frac{Q}{4\pi\epsilon}\int \frac{dr}{r^2} = \frac{Q}{4\pi\epsilon r}$$

$\therefore$ Capacitance of sphere $C = \frac{Q}{V} = 4\pi\epsilon r$.

Capacitance of concentric spheres:

Consider two concentric spheres of inner radius 'a' outer radius 'b'. ϵ_r - permittivity between medium. If charge Q is distributed over outer surface of inner Sphere, there will be equal & opposite charge induced on their outer surface on inner sphere.

Electric field intensity between two Spheres $\Big\}$ $E = \frac{Q}{4\pi\epsilon r^2}$



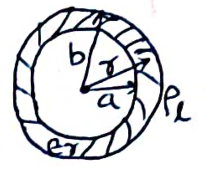
Potential difference $V = -\int_{b}^{a} \frac{Q}{4\pi\epsilon r^2}\, dr$.

$$= \frac{Q}{4\pi\epsilon}\left[\frac{1}{r}\right]_{b}^{a} = \frac{Q}{4\pi\epsilon}\left[\frac{1}{a} - \frac{1}{b}\right]$$

$$V = \frac{Q}{4\pi\epsilon}\left[\frac{b-a}{ab}\right].$$

Capacitance $C = \frac{Q}{V} = 4\pi\epsilon\left(\frac{ab}{b-a}\right)$.

Capacitance of co-axial cylinders:-



Consider a coaxial cable of inner radius 'a' & outer radius 'b'. Relative permittivity ϵ_r. Potential V is applied between cylinders. Two cylinders are charged at a rate of ρ_L C/m

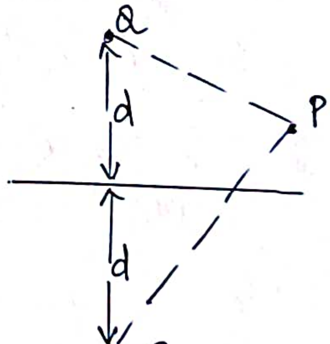
$$E = \frac{\rho_L}{2\pi\epsilon r}$$

Potential difference $V = -\int_b^a E\,dr = -\frac{\rho_L}{2\pi\epsilon}\int_b^a \frac{dr}{r}$.

$$V = \frac{\rho_L}{2\pi\epsilon} \ln\left(\frac{b}{a}\right).$$

$\therefore$ Capacitance of a coaxial cable $C = \frac{\rho_L}{V} = \frac{2\pi\epsilon}{\ln(b/a)}$ F/m.

Method of Images:

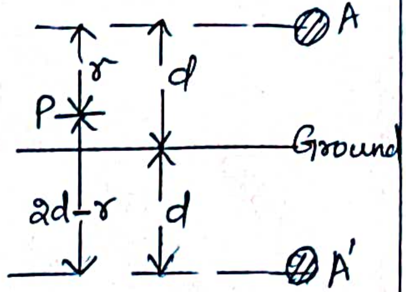


Consider a point charge Q at a distance 'd' from grounded conducting plane. This positive charge induces another charge -Q in opposite direction. This induced charge is referred as electrical image. So, this method is called as method of images.

Capacitance of infinite single wire Transmission Line:

Consider a infinite single wire transmission line A parallel to ground at a height 'd' from ground and its induced image A' at a height below the ground.

If transmission line A has charge ρ_L C/m, induces $-\rho_L$ C/m on image line A'. Electric field intensity at any point P' is algebraic sum of electric field by A & A'.



$$E = \frac{\rho_L}{2\pi \epsilon r} + \frac{\rho_L}{2\pi \epsilon (2d-r)}$$

$$= \frac{\rho_L}{2\pi \epsilon}\left(\frac{1}{r} + \frac{1}{2d-r}\right)$$

Potential difference $V = -\int E\,dr$.

$$V = -\frac{\rho_L}{2\pi \epsilon}\int_{2d-a}^{a}\left(\frac{1}{r} + \frac{1}{2d-r}\right)dr$$

$$= -\frac{\rho_L}{2\pi \epsilon}\left[\ln\frac{a}{2d-a} + \ln\frac{a}{2d-a}\right]$$

$$V = \frac{\rho_L}{\pi \epsilon}\ln\left(\frac{2d-a}{a}\right)$$

Since ground is exactly mid-way between transmission lines A & A', potential of actual line is one half of potential difference between transmission line A and image line A' (ie) V/2.

$$C = \frac{\rho_L}{V/2} = \frac{\rho_L}{\frac{\rho_L}{2\pi\epsilon}\ln\left(\frac{2d-a}{a}\right)} = \frac{2\pi\epsilon}{\ln\left(\frac{2d-a}{a}\right)} \text{ F/m}.$$

Poisson's and Laplace Equations:

Based on Gauss law in point form, divergence of electric flux density is equal to volume charge density.

$$\nabla \cdot D = \rho_v$$

$$\nabla \cdot \epsilon E = \rho_v \qquad [\because D = \epsilon E]$$

$$\nabla \cdot E = \frac{\rho_v}{\epsilon}$$

$$\nabla \cdot (-\nabla V) = \rho_v/\epsilon \qquad [\because E = -\nabla V]$$

$$\boxed{\nabla^2 V = -\rho_v/\epsilon}$$ This is Poisson's Equation.

For cartesian co-ordinates system,

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -\frac{\rho_v}{\epsilon}.$$

For cylindrical co-ordinate system,

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\left(\frac{\partial^2 V}{\partial \phi^2}\right) + \frac{\partial^2 V}{\partial z^2} = -\frac{\rho_v}{\epsilon}$$

For Spherical co-ordinate system,

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} = -\frac{\rho_v}{\epsilon}$$

If volume charge density (ρ_v) is zero, then

$\boxed{\nabla^2 V = 0}$ This is Laplace equation.

∇^2 - Laplacian operator.

Solution of Laplace Equation:

General procedure for solving a boundary value problem, using Poison's or Laplace equation is

1. Equation is solved using either
   - direct integration when V is function of one variable
   - Seperation of Variables if V is function of one or more Variables
2. The unknown integration constants are found by applying boundary conditions; so solution be unique.
3. To obtain V, Use relation : $E = -\nabla V$
4. Charge induced on conductor $Q = \int_s \sigma \, ds$
5. Capacitance is obtained from $C = Q/V$.

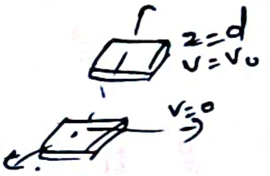
Solution of Laplace equations in Cartesian Coordinates:

Parallel Plate electrode System:

Ex: The region between two conducting plates at $x=0$ & $x=d$ is filled with perfect dielectric permittivity ϵ. If plate at $x=d$ is maintained at a voltage V_0 & at $x=0$ is grounded, find the potential distribution between plates.

By Laplace equation in cartesian co-ordinates

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 .$$



As $\frac{\partial^2 V}{\partial y^2} = \frac{\partial^2 V}{\partial z^2} = 0$, & V varies with x only, then the partial derivative becomes ordinary

$$\therefore \frac{d^2 V}{dx^2} = 0 .$$ Integrating by twice,

$$V = Ax + B .$$

Apply boundary conditions: i) At $x=0$, $V=0$ $\therefore B=0$

ii) At $x=d$, $V=V_0$, $\therefore A = \frac{V_0}{d}$

Potential distribution between parallel plate electrode system $\Big\}$ $V = \left(\frac{V_0}{d}\right)x$.

Solution of Laplace equation in Cylindrical Co-ordinates:

Potential of co-axial cable:

ex: Using Laplace equation, find potential distribution within coaxial cable of length L has an inner conductor radius 'a' & outer conductor radius 'b', if potential V_0 is applied at two conductors. Also determine E.

Laplace equation $\nabla^2 V = 0$.

In cylindrical co-ordinates

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0.$$

Since potential variation is only in radial direction, the equation reduces to

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) = 0.$$

$$\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) = 0.$$

Integrating $r\frac{\partial V}{\partial r} = A$; $\therefore \frac{\partial V}{\partial r} = \frac{A}{r}$.

$$\therefore V = A \ln r + B$$

Apply boundary conditions:
1) At $r=a$, $V=V_0$, $\therefore V_0 = A\ln a + B$
2) At $r=b$, $V=0$, $\therefore 0 = A\ln b + B$.

Solving $A = \frac{V_0}{\ln(a/b)}$ & $B = -A\ln b = -\frac{V_0 \ln b}{\ln(a/b)}$

Potential distribution $V = \frac{V_0 \ln r}{\ln(a/b)} - \frac{V_0 \ln b}{\ln(a/b)} = \frac{V_0 \ln(r/b)}{\ln(a/b)}$

Field intensity $E = -\nabla V = -\left(\frac{\partial V}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial V}{\partial \phi}\hat{a}_\phi + \frac{\partial V}{\partial z}\hat{a}_z\right)$

Potential is function of r only, then

$$\bar{E} = -\nabla V = -\frac{\partial V}{\partial r}\hat{a}_r = -\frac{V_0}{\ln(a/b)}\frac{\partial}{\partial r}\left[\ln\left(\frac{r}{b}\right)\right]\hat{a}_r$$

$$= -\frac{V_0}{\ln(a/b)} \times \frac{b}{r} \times \frac{1}{b}\hat{a}_r = \frac{V_0}{r\ln(b/a)}\hat{a}_r.$$

$$\bar{E} = \frac{V_0}{r\ln(b/a)}\hat{a}_r.$$

Solution of Laplace equation in Spherical Coordinates:

Potential of Spherical Shell electrode System:

ex: Develop expressions for the potential difference and field intensity at any point between Spherical Shells in terms of applied potential. Given: $V=V_0$ at $r=a$ & $V=0$ at $r=b$. $(a<b)$ Find $\bar{E}$.

Laplace equation in Spherical Coordinates

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}=0$$

By symmetry, field & potential depends on radial distance 'r' only.

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)=0 \quad (\text{or}) \quad \frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)=0.$$

Integrating $\frac{\partial V}{\partial r}=\frac{A}{r^2}$; $V=-\frac{A}{r}+B$

Apply boundary Conditions:

1) At $r=a$, $V=V_0$, $\therefore V_0=-\frac{A}{a}+B$

2) At $r=b$, $V=0$, $\therefore 0=-\frac{A}{b}+B$.

Solving, $A=\frac{V_0\,ab}{a-b}$ & $B=\frac{V_0\,a}{a-b}$.

Potential $V=-\left(\frac{V_0\,ab}{a-b}\right)\frac{1}{r}+\frac{V_0\,a}{a-b}=\frac{-V_0\,ab}{a-b}\left(\frac{1}{r}-\frac{1}{b}\right)$

$$V=\frac{V_0\,a}{r}\left(\frac{b-r}{b-a}\right)$$

field intensity $E=-\nabla V=-\frac{\partial V}{\partial r}\hat{a_r}=\frac{-V_0\,ab}{b-a}\left[\frac{\partial}{\partial r}\left[\frac{1}{r}-\frac{1}{b}\right]\hat{a_r}\right]$

$$\bar{E}=-\left(\frac{V_0\,ab}{a-b}\right)\frac{1}{r^2}\,\hat{a_r}.$$

Application of Laplace Equation:

Isolated Conducting Sphere:

Consider the applications of Laplace equation to an isolated sphere of radius 'a' with uniformly distributed charge Q. It is in a medium of permittivity ϵ_0. To solve Laplace equation outside sphere which gives potential distribution satisfies boundary conditions (i.e) potential is constant over spherical surface & zero at infinity.

V is a function of R alone.

$$\therefore \nabla^2 V = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2 \frac{\partial V}{\partial R}\right) = 0.$$

// By expanding, $\frac{V}{R^2} + \frac{2}{R}\frac{\partial v}{\partial R} = 0$ [2nd order differential equation]

This gives general expression for potential as a function of R.

$$\frac{\partial}{\partial R}\left(R^2 \frac{\partial v}{\partial R}\right) = 0 \quad ; \quad R^2 \frac{\partial v}{\partial R} = c_1 \text{ (constant)}.$$

$$\frac{\partial v}{\partial R} = \frac{c}{R^2}, \quad \therefore V = -\frac{c_1}{R} + c_2$$

In general, both constants c_1 & c_2 are determinable from boundary conditions.

At infinite distance, $V = 0$, $\therefore$ equation reduces to

$$V = -\frac{c_1}{R} \quad \text{—(1)}$$

Generally, $V = \frac{Q}{4\pi\epsilon_0 R}$ for sphere —(2)

By comparing (1) & (2), $c_1 = -\frac{Q}{4\pi\epsilon_0}$ [$\because R \geq a$].

Uniqueness of Electro Static Solutions:

Any solution of Poisson's and Laplace's equation which satisfies the same boundary conditions must be the only solutions regardless of the method used. This is known as Uniqueness theorem. It applies to any solution of Poisson's and Laplace's equation in a given region.

Uniqueness Theorem: If a solution to Laplace's equation can be found that satisfies the boundary conditions, then the solution is unique.

Let us, assume that there are two solutions V_1 & V_2 of Laplace's equation both of which satisfy the prescribed boundary conditions.

(ie) $\nabla^2 V_1 = 0$ & $\nabla^2 V_2 = 0$

Based on boundary condition, $V_1 = V_2$

Consider their difference

$$V_d = V_2 - V_1$$

which obeys $\nabla^2 V_d = \nabla^2 V_2 - \nabla^2 V_1 = 0.$

Based on boundary condition

$$V_d = 0.$$

Electro Static Energy:

The capacitor stores electro static energy equal to work done to build up charge. If voltage source is connected across capacitor, it charges.

Potential $V = \frac{dW}{dQ}$; $\therefore dW = V.dQ$

$dW = \frac{Q}{C} dQ$ $[\because V = \frac{Q}{C}]$

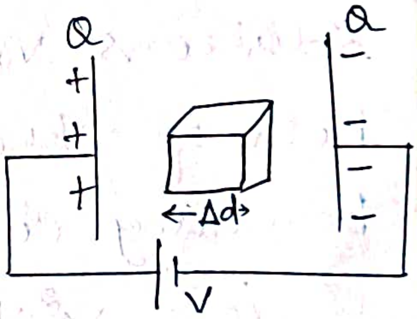
Capacitor is charged to the value of Q.

Total work done $W = \int_0^Q \frac{Q}{C} dQ = \frac{1}{C}\left[\frac{Q^2}{2}\right]_0^Q = \frac{Q^2}{2C}$ Joules

But $Q = CV$, $W = \frac{1}{2} CV^2 = \frac{1}{2} QV$ joules.

Energy Density:

Consider a elementary cube of side Δd parallel to the plates of a capacitor.



Capacitance of elemental capacitor $\Big\}$ $\Delta C = \frac{\epsilon A}{\Delta d} = \frac{\epsilon (\Delta d)^2}{\Delta d}$.

$\Delta C = \epsilon \Delta d$

Energy stored in elemental capacitor is

$\Delta W = \frac{1}{2} \Delta C (\Delta V)^2$.

Potential difference $\Delta V = E.\Delta d$

$\therefore$ Total stored energy $\Delta W = \frac{1}{2}(\epsilon \Delta d)(E.\Delta d)^2$

$= \frac{1}{2} \epsilon E^2 (\Delta d)^3$

$= \frac{1}{2} \epsilon E^2 \Delta v$

where $\Delta v = (\Delta d)^3$ is elementry volume.

$\therefore$ Energy density $\frac{\Delta W}{\Delta v} = \frac{1}{2} \epsilon E^2 = \frac{1}{2} D.E$ joules/m^3. $[\because D = \epsilon E]$

Current Density and Ohm's Law:

Conduction Current :

The conduction current I is flowing through a conductor whose resistance is R when potential V is applied across the conductor.

Conduction current $I_c = V/R$ (Ohm's Law)

Resistance of conductor $R = \frac{\rho l}{A}$; ρ – resistivity, l – length of conductor

$R = \frac{l}{\sigma A}$, $[\because \sigma = 1/\rho$, σ – Conductivity$]$; A – Area.

Potential $V = El$ where E – Electric field.

$\therefore$ Conduction current $I_c = \frac{V}{R} = \frac{El}{l} \cdot \sigma A = E\sigma A$

Conduction current density $J_c = \frac{I_c}{A} = \sigma E$

$\boxed{J = \sigma E}$ → Point form of Ohm's law.

Displacement Current :

The displacement current I_D is flowing through a capacitor when ac voltage is applied across the capacitor.

$$I_D = \frac{dQ}{dt} = C\frac{dV}{dt} \quad [\because Q = CV]$$

For the parallel plate capacitor

$C = \frac{\epsilon A}{d}$; ϵ – Permittivity of medium
A – Area of plate
d – distance between two plates.

Displacement current $I_D = \frac{\epsilon A}{d} \cdot \frac{dV}{dt}$

$$= \epsilon A \cdot \frac{dE}{dt} = A\frac{dD}{dt} \quad \left[\because V = Ed, \; D = \epsilon E\right]$$

Displacement Current density $J_D = \frac{I_D}{A} = \frac{\partial D}{\partial t}$.

Continuity Equation:

Current is defined as rate of movement of charge passing a given reference point.

$$I = dQ/dt$$

The increment of current ΔI crossing an increment surface ΔS is $\Delta I = J \cdot \Delta S$ where J - current density

Total current $I = \int_S J \cdot ds$

Current through closed surface is $I = \oint_S J \cdot ds$ and this outward flow of positive charge must be balanced by a decrease of positive charge within the closed surface. If charge inside the closed surface is Q, then rate of decrease is $\frac{dQ}{dt}$ and the principle of conservation of charges requires

$$I = \oint J \cdot ds = -\frac{dQ}{dt}$$; This is integral form of continuity equation.

The differential or point form is obtained by changing the surface integral to a volume integral by using divergence theorem

$$\oint_S J \cdot ds = \int_V (\nabla \cdot J)\, dv$$

Then, $\int_V (\nabla \cdot J)\, dv = -\frac{dQ}{dt} = -\frac{d}{dt}\int_V \rho_v \, dv$ $\left[\because Q = \int_V \rho_v \, dv\right]$

$$\int_V \nabla \cdot J \, dv = -\int \frac{\partial \rho_v}{\partial t} \cdot dv$$

$$\nabla \cdot J = -\frac{\partial \rho_v}{\partial t}$$; This is point or differential form of continuity equation.

Electromotive Force & Kirchoff's Voltage law:

For Static electric field, $\oint E \cdot dl = 0$

Point form of ohm's law, $J = \sigma E$

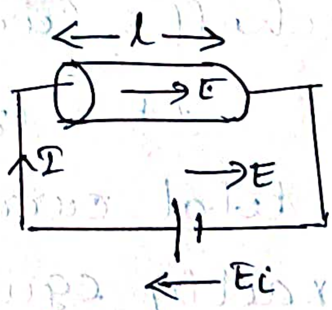
$\therefore E = J/\sigma$

$$\therefore \oint \frac{J}{\sigma} \cdot dl = 0.$$

An electro-static field can't maintain a steady current in a closed circuit. Motion of charged carriers in a circuit which is required to estabilish a steady current is a dissipative process.

Moving charges give neither gain nor loss energy after completing one trip. Loss of energy is supplied by sources of non-conservative field.

Consider a battery, chemical action causes accumulation of positive & negative charges, these estabilish field E both inside & outside battery.



When battery is under open circuit, no current flows & force acts on charges must be zero.

$$E_i + E = 0$$

Under closed circuit, $\frac{J}{\sigma} = E_{tot} = E_i + E$.

Conservative field $E . dL = 0$.

Since, E_i is zero, outside battery, finite conductivity only in conductor region.

$$\int E_i \cdot dl = \frac{J}{\sigma} l \qquad \left[J = \frac{I}{A}\right]$$

$$= \frac{1}{\sigma}\left(\frac{I}{A}\right) l \qquad \left[\frac{1}{\sigma} = \rho\right.$$

$$= \frac{\rho l}{A} I = RI \qquad \left. R = \frac{\rho L}{A}\right]$$

Kirchoff's Voltage Law:

It states that for a closed loop series path, the algebraic sum of all the voltages around any closed loop in a circuit is equal to zero.

Kirchoff's Current Law:

It states that for a parallel path the total current entering a circuit junction is exactly equal to the total current leaving the same junction.

Problems:

1) Radii of two spheres differ by 4 cm & capacity of spherical capacitor is 53.33 pf. Calculate radii assume air as dielectric.

Given $b - a = 4\times10^{-2}\,m$, $C = 53.33\times10^{-12}\,F$

$$C = 4\pi\epsilon_0\left(\frac{ab}{b-a}\right).$$

$$53.33\times10^{-12} = 4\pi\times8.85\times10^{-12}\left(\frac{ab}{4\times10^{-2}}\right)$$

$$\therefore ab = 192\times10^{-4}.$$

From ab & $b-a$, $a = 12$ cm, $b = 16$ cm.

2) If concentric cable condenser, diameter of inner and outer cylinders are 3 mm & 10 mm respectively. If ϵ_r for insulation is 3, find its capacity/mtr.

$$C = \frac{2\pi\epsilon_0\epsilon_r}{\ln b/a} = \frac{2\pi\times8.85\times10^{-12}\times3}{\ln 10/3} = 1.384\times10^{-10}\,F/m.$$

3) If a parallel plate capacitor consists of two plates each 30 cm x 30 cm spaced 2 mm apart & two dielectrics each 1 mm thickness having $\epsilon_r = 3$ & 5. If potential difference between plates is 5000 V, calculate voltage gradient in each dielectric.

$$V_1 + V_2 = 5000$$

$$V_1 = E_1(t-t_1) = E_1(2-1)\times10^{-3} = E_1\times10^{-3}$$

$$V_2 = E_2 t_1 = E_2\times10^{-3}$$

$$\therefore (E_1+E_2)10^{-3} = 5000$$

[$t = 2$ mm
$t_1 = 1$ mm
$\epsilon_{r_1} = 3$
$\epsilon_{r_2} = 5$]

$$E_1 = \frac{D}{\epsilon_0 \epsilon_{r_1}} = \frac{D}{3\epsilon_0} \quad ; \quad E_2 = \frac{D}{\epsilon_0 \epsilon_{r_2}} = \frac{D}{5\epsilon_0}.$$

$$\therefore 3\epsilon_0 E_1 = 5\epsilon_0 E_2$$

$$3/5\, E_1 = E_2.$$

$$\text{So, } (E_1 + 3/5\, E_1)10^{-3} = 5000$$

$$8/5\, E_1 = 5 \times 10^6$$

$$\therefore E_1 = 3125 \text{ kV/m}, \quad E_2 = 1875 \text{ kV/m}.$$

4) Calculate conductivity of 4mm diametee, 5m long wire if its measured resistance is 12 mΩ.

$$\sigma = \frac{W}{RA} = \frac{5}{12 \times 10^{-3} \pi (2 \times 10^{-3})^2} = 3.3 \times 10^7 \text{ ℧/m}.$$

5) A flat slab of Sulphur ($\epsilon_r = 4$) is placed normal to uniform field. If polarisation charge surface density σ_p on slab surface is 1 C/m². Determine flux density & field intensity of a slab.

$$1 + \frac{\chi}{\epsilon_0} = \epsilon_r = 4 \quad ; \quad \therefore \chi = 3 \times 8.85 \times 10^{-12} = 2.655 \times 10^{-11}$$

$$P = \sigma_p = \chi E_i \quad ; \quad \therefore E_i = 1/2.655 \times 10^{-11} = 3.766 \times 10^{10} \text{ V/m}.$$

$$\bar{D} = \epsilon_0 E_i + P \quad ; \quad \therefore \bar{D} = 8.85 \times 10^{-12} \times 3.76 \times 10^{10} + 1 = 1.33 \text{ C/m}^2$$

6) Find the resistance of 1km length of Si wire which has a conductivity $\sigma = 6.17 \times 10^{-7}$ ℧/m and $r = 1 \times 10^{-3}$ m.

$$\text{Resistance } R = \frac{\rho L}{A} = \frac{l}{\sigma A} \quad ; \quad A = \pi r^2 = \pi \times 10^{-6} \text{ m}$$

$$\therefore R = \frac{1 \times 10^3}{6.17 \times 10^{-7} \times \pi \times 10^{-6}} = 5.159 \,\Omega.$$

UNIT III . MAGNETO STATICS

Magnetic Force on a moving charge: (Lorentz Force Equation)

A charged particle in motion in a magnetic field of flux density 'B' is experienced a force. Force is proportional to the product of magnitude of charge Q, its velocity v and flux density B and to the sine of the angle between v and B.

$$F = Q(\bar{V} \times \bar{B}) = QvB \sin\theta .$$

The electrical force on a charged particle in electric field of intensity E is $F = QE$

The force on a moving particle due to combined electric and magnetic field is obtained.

$$F = Q\bar{E} + Q\,\bar{V}\times\bar{B} = Q[\bar{E} + \bar{V}\times\bar{B}]$$

Force experienced by test charge is called Lorentz force. It is maximum if direction of movement of charge is perpendicular to orientation of field lines.

Force on a current element:

The force on a current element is due to charge element, dQ as

$$dF = dQ\,(v \times B)$$

$$= dQ\left(\frac{dl}{dt} \times B\right) \quad \left[\because \text{velocity } v = \frac{dl}{dt}\right].$$

$$= \frac{dQ}{dt}\,(dl \times B) \quad \left[\because \text{Current } I = \frac{dQ}{dt}\right]$$

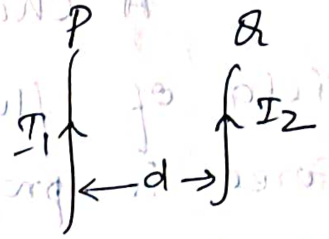
$$dF = I\,dl \times B$$

$\therefore$ Force $F = (I \times B)\, l$

Magnitude of force is $F = BIl \sin \theta$.

Force between current elements :

Consider two current carrying Conductors Seperated by a distance 'd'.



Consider conductor P produces magnetic field whose flux density B at conductor Q

$$B = \frac{\mu_0 I}{2\pi d}.$$

Force on conductor Q due to P is $F = BI_2 l$.

where l – length of conductor

$$F = \frac{\mu_0 I_1 I_2 l}{2\pi d}.$$

If currents are flowing in same direction, there is force of attraction.

If currents are flowing in opposite direction, there is a force of repulsion. However, the value will be same.

$$F = \frac{\mu_0 I_1 I_2 l}{2\pi d} \; N.$$

If conductors are infinitely long, force per unit length is $F = \frac{\mu_0 I_1 I_2}{2\pi d}$ N/m.

ex: Determine the force per metre length between two long parallel wires seperated by 5 cm in air and carrying currents of 40 A. a) in same direction. b) in opposite direction.

$$F = \mu_0 \frac{I_1 I_2}{2\pi d} = \frac{4\pi \times 10^{-7} \times 40 \times 40}{2\pi \times 5 \times 10^{-2}} = 6.4 \times 10^{-3} \text{ Nw}.$$

Magnitude same, but (a) is attractive (b) is repulsive.

UNIT III. STATIC MAGNETIC FIELDS (MAGNETO STATICS)

Electric charge at rest produces an electric field called electrostatic field. Electric charges in motion that is current produce magnetic field. The source of steady magnetic field maybe a permanent magnet, an electric field changing linearly with time. The magnetic flux lines are enclosed around the element.

Magnetic Field Intensity:

The magnetic field at a given point is specified by both direction and magnitude and is described by magnetic field intensity (H)

Magnetic field intensity is defined as the force experienced by a north pole of one weber placed at that point. (or) the magnetomotive force per unit length produced by the steady current in a magnetic circuit.

It's unit is Newton/weber (N/Wb) (or) Ampere per metre (A/m) (or) Ampere-turn per metre (AT/m).

Magnetic Flux & Magnetic flux density:

Magnetic flux is the group of magnetic field lines emitted outward from the north-pole of a magnet. It's unit is Weber.

Magnetic flux density ($\bar{B}$) is defined as magnetic flux passing per unit area. It's unit is weber/m^2 (or) Tesla.

$$B = \phi/A \text{ tesla}; \quad \text{Also, } B = \mu H.$$

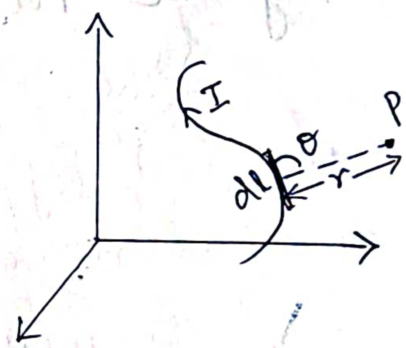
where $\mu = \mu_0 \mu_r \rightarrow$ permeability of medium.

$\mu_0 = 4\pi \times 10^{-7}$ (H/m) $\rightarrow$ Free space permeability.

μ_r = relative permeability.

H = magnetic field intensity.

Biot - Savart Law:



The magnetic flux density produced by a current element at any point in a magnetic field is proportional to the current element and inversely proportional to square of the distance between them.

The magnetic flux density at any point P due to current element Idl is given by

$$dB \propto \frac{I\,dl}{r^2}$$

$$dB = \frac{\mu I\,dl \times \hat{u}_r}{4\pi r^2}$$ where μ - permeability of medium

$I\,dl$ - current element

r - distance between point P and current element.

$\hat{u}_r$ - unit vector.

Expression indicates direction of dB, contribution of current element in producing field at point P, field acts in direction perpendicular to plane containing element & line joining the element to P given by cross product $dl \times \hat{u}_r$.

Magnitude of magnetic flux density $dB = \frac{\mu I \, dl \sin\theta}{4\pi r^2}$.

Magnetic field intensity is $dH = \frac{I dl}{4\pi r^2} \hat{u}_r$.

Its magnitude is $dH = \frac{I dl \sin\theta}{4\pi r^2}$; This is referred as ampere's law for current element

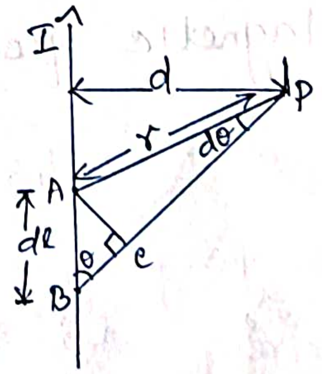
Resultant field at P due to whole conductor is given by adding up contributions of current elements.

Flux density at Point 'P' is $B_P = \oint \frac{\mu I}{4\pi r^2} dl \times \hat{u}_r$

Field intensity at 'P' is $H_P = \oint \frac{I}{4\pi r^2} dl \times \hat{u}_r$.

Magnetic field intensity at any point due to infinitely straight conductor:

Consider a infinitely straight conductor carrying current I and also consider current element Idl. Let 'P' be any point at which magnetic field intensity is to be measured at a distance 'r' from the current element Idl.



By Biot Savart law, Magnetic flux density at any point 'P' is $B = \frac{\mu_0 I}{4\pi} \int \frac{dl \sin\theta}{r^2}$ $[\because \mu_r = 1]$

From ΔABC, $\frac{AC}{AB} = \sin\theta$.

$AC = dl \sin\theta$, since $AB = dl$.

But arc $AC = r d\theta$;

$\therefore dl \sin\theta = r d\theta$; $\frac{dl \sin\theta}{r} = d\theta$

Substitute this value in equation 'B'.

$$B = \frac{\mu_0 I}{4\pi} \int \frac{d\theta}{r} ;$$

From fig, $d/r = \sin\theta$, $\therefore r = \frac{d}{\sin\theta}$.

So, $B = \frac{\mu_0 I}{4\pi d} \int_0^{\pi} \sin\theta \, d\theta = \frac{\mu_0 I}{4\pi d} [-\cos\theta]_0^{\pi}$

$= \frac{\mu_0 I}{4\pi d} \times 2 = \frac{\mu_0 I}{2\pi d}$.

Magnetic flux density due to infinite conductor

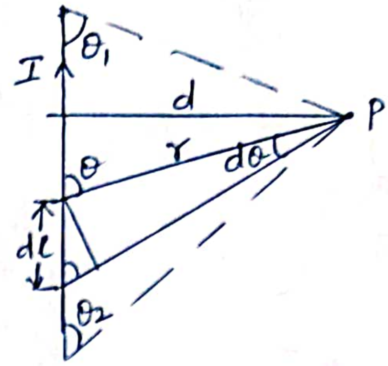
$$B = \frac{\mu_0 I}{2\pi d} \text{ wb/m}^2 .$$

Magnetic field intensity $H = \frac{I}{2\pi d}$ A/m.

Magnetic field intensity due to finite length Conductor:

Consider a conductor of finite length carrying current I and consider a small current element Idl in the conductor at a distance 'r' from the point P where magnetic field is to be determined.

By Biot Savart law, flux density at P due to current element Idl is $dB = \frac{\mu_0 I dl \sin\theta}{4\pi r^2}$.



From fig. $r\,d\theta = dl \sin\theta$ (arc).

Substitute this in dB equation

$$dB = \frac{\mu_0 I\, r\,d\theta}{4r^2\pi} = \frac{\mu_0 I\, d\theta}{4\pi r}$$

From fig. $\frac{d}{r} = \sin\theta$, $r = \frac{d}{\sin\theta}$

Substitute value of r, $dB = \frac{\mu_0 I \sin\theta\, d\theta}{4\pi d}$

$\therefore$ Total flux density $B = \frac{\mu_0 I}{4\pi d}\int_{\theta_1}^{\pi-\theta_2} \sin\theta\, d\theta$.

$$B = \frac{\mu_0 I}{4\pi d}\left[-\cos\theta\right]_{\theta_1}^{\pi-\theta_2} = \frac{\mu_0 I}{4\pi d}\left[\cos\theta_1 + \cos\theta_2\right]$$

If it is infinitely long, $\theta_1 = \theta_2 = 0$.

Then, $B = \frac{\mu_0 I}{4\pi d} \times 2 = \frac{\mu_0 I}{2\pi d}$. for infinitely long conductor.

Similarly, $H = \frac{I}{2\pi d}$ A/m for infinite conductor.

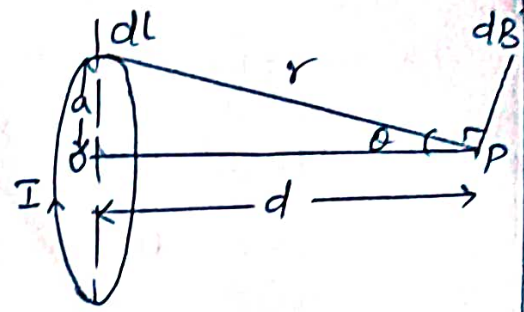
Magnetic field intensity at any point along the axis of circular conductor:

Consider a circular coil of radius 'a' carrying current I and also consider current element Idl. Let P be any point at a distance d from the centre of the coil.

Magnetic flux density at P due to current element

$$dB = \frac{\mu_0 I\, dl}{4\pi r^2}.$$

From ΔAOP, $r^2 = a^2 + d^2$.



$$dB = dB \cos\theta$$
$$= dB \cdot \frac{a}{\sqrt{a^2+d^2}}$$

$$\therefore dB = \frac{\mu_0 I dl}{4\pi(a^2+d^2)} \cdot \frac{a}{\sqrt{a^2+d^2}} = \frac{\mu_0 a I dl}{4\pi(a^2+d^2)^{3/2}}.$$

Magnetic flux density due to circular coil is

$$B = \frac{\mu_0 a I}{4\pi(a^2+d^2)^{3/2}} \int dl$$

$$= \frac{\mu_0 a I}{4\pi(a^2+d^2)^{3/2}} \cdot 2\pi a = \frac{\mu_0 a^2 I}{2(a^2+d^2)^{3/2}} \text{ wb/m}^2.$$

Magnetic field intensity $H = \frac{I a^2}{2(a^2+d^2)^{3/2}}$ A/m.

If $d=0$, the field at the centre

Flux density $B = \frac{\mu_0 I a^2}{2a^3} = \frac{\mu_0 I}{2a}$ wb/m^2

Field intensity $H = \frac{I}{2a}$ A/m.

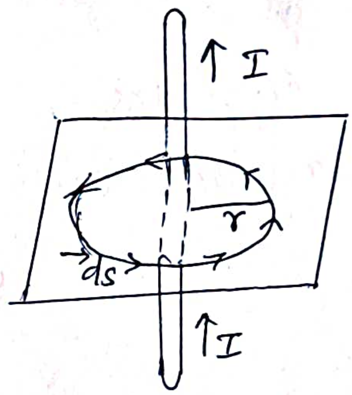
Ampere's Circuital Law:

This law states that the line integral of the magnetic field intensity ($\bar{H}$) around any closed path is equal to direct current enclosed by the path.

$$\oint H \cdot dl = I.$$

Proof: Consider an infinite straight conductor carrying a steady current I along the z-direction. Also, consider a closed path of radius 'r' enclosing the conductor.

By Biot Savart law, magnetic field intensity at any point on the circle is



$$\bar{H} = \frac{I}{2\pi r}\hat{a_\phi}$$

Now, the elemental length of the circle is $\bar{dl} = r d\phi \hat{a_\phi}$

$$\oint_l H \cdot dl = \oint_l \frac{I}{2\pi r}\hat{a_\phi} \cdot (r d\phi \hat{a_\phi})$$

$$= \oint_l \frac{I}{2\pi} d\phi = \frac{I \times 2\pi}{2\pi} = I$$

$\therefore \oint_l H \cdot dl = I$. This is Ampere's Circuital Law.

Point form of Ampere's Circuital Law:

Apply stokes theorem in ampere's law, we get $\oint_l H \cdot dl = \oint_s (\nabla \times H) \cdot ds = I$ —①

But, in terms of current density, $I = \oint_s J \cdot ds$ —②

Compare ① & ②, we get.

$$\oint_s (\nabla \times H) \cdot ds = \oint_s J \cdot ds$$

$$\therefore \boxed{\nabla \times H = J}$$

This is differential or Point form of Ampere's Circuital Law.

Applications of Ampere's Law:

Ampere's law can be applied to:

1. Infinitely long straight wires carrying steady current I.
2. Infinitely large sheet of thickness b with a current density $\bar{J}$.
3. Infinite solenoid and Toroid.

ex: Consider a long straight wire of radius r carrying current I. Find magnetic field intensity?

By Ampere's law, $\oint H \cdot dl = I$

For wire, $H \cdot (2\pi r) = I$

$\therefore$ Magnetic field intensity $H = \frac{I}{2\pi r}$.

Scalar Magnetic Potential:

Ampere's law states that the line integral of the field H around a closed path is equal to the current enclosed. $\oint H \cdot dl = I$

If no current is enclosed (ie) $J = 0$.

$\therefore \oint H \cdot dl = 0$.

Magnetic field is expressed as negative gradient of a scalar function.

$H = -\nabla V_m$, where V_m is scalar magnetic potential

$V_m = -\int H \cdot dl$.

This scalar potential also satisfies Laplace equation.

In free space $\nabla \cdot B = 0.$

$$\mu_0 \, \nabla \cdot H = 0.$$

$$\mu_0 \cdot \nabla(-\nabla V_m) = 0 \qquad [\because -\nabla V_m = H]$$

$$-\mu_0 \, \nabla^2 V_m = 0$$

$$\therefore \nabla^2 V_m = 0.$$

Vector Magnetic Potential:

Scalar magnetic potential exists if there is no current enclosed. (ie) $\oint H \cdot dl = 0$. If current is enclosed, the potential which depends upon current element is no more scalar but it is vector quantity. Since, divergence of a vector is a scalar, vector potential is expressed in curl.

(ie) $\nabla \cdot B = 0.$

$B = \nabla \times A$ where A is magnetic Vector potential.

Take curl on both sides.

$$\nabla \times B = \nabla \times \nabla \times A$$

But $\nabla \times \nabla \times A = \nabla(\nabla \cdot A) - \nabla^2 A$ $\quad [\int B \cdot dl = \mu \cdot I = \iint \text{curl} B \cdot ds$

$\nabla \times B = \mu J \rightarrow$ $\quad \mu I = \iint \mu J \cdot ds$

$$\therefore \nabla(\nabla \cdot A) - \nabla^2 A = \mu J.$$

for steady dc, $\nabla \cdot A = 0$, then $-\nabla^2 A = \mu J.$

$$\nabla^2 A_x \bar{x} + \nabla^2 A_y \bar{y} + \nabla^2 A_z \bar{z} = -\mu(J_x \bar{x} + J_y \bar{y} + J_z \bar{z})$$

Equating $\nabla^2 A_x = -\mu J_x$, $\nabla^2 A_y = -\mu J_y$, $\nabla^2 A_z = -\mu J_z$.

They are in the form of Poisson's equations.

$[\nabla^2 V = -\frac{\rho}{\epsilon}, \; V = \frac{1}{4\pi\epsilon} \iiint \frac{\rho \, dv}{r}$

From above equations, magnetic vector potential canbe written as,

$$A_x = \frac{\mu}{4\pi}\int_v \left(\frac{J_x}{r}\right) dv \quad ; \quad A_y = \frac{\mu}{4\pi}\int_v \left(\frac{J_y}{r}\right) dv \; ;$$

$$A_z = \frac{\mu}{4\pi}\int_v \left(\frac{J_z}{r}\right) dv.$$

In General,

Magnectic Vector Potential $A = \dfrac{\mu}{4\pi}\iiint_v \dfrac{J}{r}\, dv.$

$B = \nabla \times A$
$J = \nabla \times H$
$\mu J = \nabla \times B$
$\nabla \cdot B = 0$
$\nabla \cdot J = 0$

Derivation of Steady Magnetic Field Laws:

All magnetic field quantities relationship is obtained from $H = \oint \dfrac{I\, dl \times \hat{u_r}}{4\pi r^2}$

In free space, $B = \mu_0 H$, & $B = \nabla \times A$.

But, $A = \int_v \dfrac{\mu_0 J}{4\pi r}\, dv$

Add subscripts to indicate the point at which the current element is located (x_1, y_1, z_1) & point at which A is given $(x_2\, y_2\, z_2)$. Differential vol. element dv is written as dv_1

Using Subscripts, $A_2 = \displaystyle\int_v \frac{\mu_0 J_1\, dv_1}{4\pi r_{12}}$; $H = \dfrac{B}{\mu_0} = \dfrac{\nabla \times A}{\mu_0}$

$$H_2 = \frac{\nabla_2 \times A_2}{\mu_0} = \frac{1}{\mu_0} \nabla_2 \times \int_v \frac{\mu_0 J_1\, dv_1}{4\pi r_{12}} = \frac{1}{4\pi}\int_v \frac{\nabla_2 \times J_1}{r_{12}}\, dv_1$$

The curl of the product of a scalar & a vector is given by identity.

$$\nabla \times (SV) \equiv (\nabla S) \times V + S(\nabla \times V)$$

$$\therefore \; H_2 = \frac{1}{4\pi}\int_v \left[\left(\nabla_2 \frac{1}{R_{12}}\right) \times J_1 + \frac{1}{R_{12}}(\nabla_2 \times J_1)\right] dv_1$$

Second term of integrand is zero, because $\nabla \times J_1$ denotes partial derivatives of $x_1, y_1, \& z_1$ with respect to $x_2, y_2 \& z_2$. The first set of var. is not a fn of 2nd set, and all partial derivatives are zero

r_{12} interms of coordinates,

$$r_{12} = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

$$\nabla_2 \frac{1}{r_{12}} = -\frac{r_{12}}{r_{12}^3} = -\frac{a_{r_{12}}}{r_{12}^2}$$; Substitute this, we get

$$H_2 = -\frac{1}{4\pi}\int_v \frac{a_{r_{12}} \times J_1}{r_{12}^2}\, dv_1$$

(or) $$H_2 = \oint_v \frac{I_1 dL_1 \times a_{r_{12}}}{4\pi r_{12}^2}$$

Now, consider Ampere's circuital law in point form

$$\nabla \times H = J.$$

$$\nabla \times H = \nabla \times \frac{B}{\mu_0} = \frac{1}{\mu_0}\nabla\times\nabla\times A = \frac{1}{\mu_0}\left[\nabla(\nabla\cdot A) - \nabla^2 A\right]$$

Find the div. of A by applying divergence operation

$$\nabla_2 \cdot A_2 = \frac{\mu_0}{4\pi}\int_v \nabla_2 \cdot \frac{J_1}{r_{12}}\, dv_1$$

Using vector identity $\nabla\cdot(SV) \equiv V\cdot(\nabla S) + S(\nabla\cdot V)$.

Thus, $$\nabla_2\cdot A_2 = \frac{\mu_0}{4\pi}\int_v \left[J_1\left(\nabla_2 \frac{1}{R_{12}}\right) + \frac{1}{R_{12}}(\nabla_2\cdot J_1)\right] dv_1$$

Second part of integrand is zero; we know that

$$\nabla_2 (1/R_{12}) = \frac{-R_{12}}{R_{12}^3}.$$

$$\therefore \nabla_1 \frac{1}{r_{12}} = \frac{r_{12}}{r_{12}^3} = -\nabla_2 \frac{1}{r_{12}}$$

Thus, $$\nabla_2\cdot A_2 = \frac{\mu_0}{4\pi}\int_v \left[-J_1\cdot\left(\nabla_1 \frac{1}{r_{12}}\right)\right] dv_1$$, vector identity applied again

$$= \frac{\mu_0}{4\pi}\int_v \left[\frac{1}{r_{12}}\overset{\to 0}{(\nabla_1\cdot J_1)} - \nabla_1\cdot\left(\frac{J_1}{r_{12}}\right)\right] dv_1$$ with steady mag field continuity eqn shows $\nabla\cdot J \to 0$

Apply div. theorem to 2nd term.

$$\nabla_2\cdot A_2 = -\frac{\mu_0}{4\pi}\oint_{S_1} \frac{J_1}{r_{12}}\, ds_1.$$ $$\therefore A = \int_v \frac{\mu_0 J_x\, dv}{4\pi r}$$

From poison's equation $\nabla^2 V = -\rho/\epsilon_0$.

Similarly, we have $\nabla^2 A_x = -\mu_0 J_x$, $\nabla^2 A_y = -\mu_0 J_y$, $\nabla^2 A_z = -\mu_0 J_z$

$$\boxed{\nabla^2 A = -\mu_0 J}$$

If $J=0$, $\nabla^2 A = 0$. (or)

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial A_z}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 A_z}{\partial \phi^2} + \frac{\partial^2 A_z}{\partial z^2} = 0.$$

A is function of ρ only, then $\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial A_z}{\partial \rho}\right) = 0$.

$\therefore A_z = c_1 \ln \rho + c_2$.

Choose zero reference at $\rho = b$, then $A_z = c_1 \ln \frac{\rho}{b}$.

Take $B = \nabla \times A = -\frac{\partial A_z}{\partial \rho} a_\phi = -\frac{c_1}{\rho} a_\phi = B$.

$\therefore H = -\frac{c_1}{\mu_0 \rho} a_\phi$.

Evaluate integral $\oint H \cdot dl = I = \int_0^{2\pi} -\frac{c_1}{\mu_0 \rho} a_\phi \cdot \rho \, d\phi \, a_\phi = \frac{-2\pi c_1}{\mu_0}$

Thus, $c_1 = -\frac{\mu_0 I}{2\pi}$ (or) $A_z = \frac{\mu_0 I}{2\pi} \ln \frac{b}{\rho}$.

$$H_\phi = I/2\pi\rho.$$

Law of no magnetic monopoles:

It states that magnetic field B has divergence equal to zero. (ie) it is a solenoidal vector field. It is equal to the statement that magnetic monopoles don't exist.

ie $\nabla \cdot B = 0$.

$\nabla \rightarrow$ Differential Vector Operator

$B \rightarrow$ Magnetic flux density.

Force and Torque on a closed Circuit:

Consider the rectangular loop abcd with dimensions $l \times b$ carrying current I is situated in a magnetic field. Sides bc & da are parallel to the field experiences no force. Equal forces act on length l in opposite direction. Direction of force acts on ab is that it will tend to move downwards & CD moves upwards.

Tangential force multiplied by radial distance at which it acts is called Torque (or) mechanical moment on the loop.

Force acting on loop $F = BIl \sin\theta$.

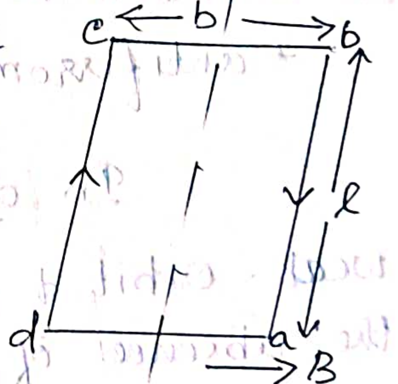
Total torque on the loop T = 2 × torque on each side

= 2 × force × distance.

$= 2 \times BIl \times \frac{b}{2} \sin\theta$

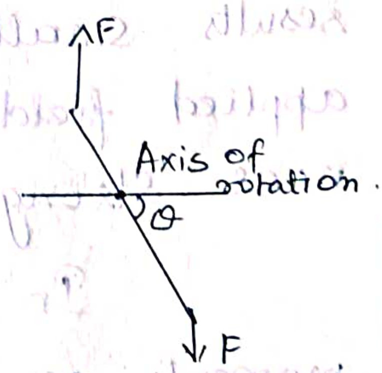
$= BI(l \times b) \sin\theta$

$T = BIA \sin\theta$.



Axis of rotation.

Torque increases with current & flux density.



Magnetic moment of loop $m = IA$.

This is a vector with direction given by unit normal to the plane of loop.

$m = IA\,\hat{n}$

$\therefore T = mB \sin\theta\,\hat{n}$.

In vector form, Torque $T = m \times B$.

when $\theta = 90°$, $T = mB$, $\therefore m = T/B$

Magnetic moment is defined as maximum torque on loop per unit magnetic induction.

The nature of Magnetic Materials:

Magnetic field is present around a current carrying conductor. All materials show magnetic effects. Some of them are so weak. They are non magnetic. (Vaccum)

Depends on magnetic behaviour, substances can be classified into three groups. They are

1. diamagnetic 2. Paramagnetic 3. Ferromagnetic

In addition to this, we have

* antiferromagnetic * ferrimagnetic * Superparamagnetic.

In diamagnetic, magnetic effects are weak. Orbit & Spin magnetic moments cancel in the absence of external field. Applied field causes Spin moment to slightly exceed the orbital moment, results small net magnetic moment which opposes applied field. Thus, if diamagnetic is brought near strong magnet, it is repelled.

In other materials, orbit & spin magnetic moments are unequal, results in net magnetic moment with no applied field. Random orientation of atoms result in little net magnetic moment. Internal interactions & thermal agitation tend to inhibit the process. They are having significant magnetic effects. They are paramagnetic. When it is brought near strong magnet, it is attracted.

In few materials - iron, ni, co, - a special phenomenon occurs which facilitates alignment process. They are ferromagnetic. Quantum effect between adjacent atoms locks their magnetic moments into rigid parallel configurations over regions - domains which contain many atoms. At above critical temperature, quantum effect disappears & material reverts to an ordinary paramagnetic.

In antiferro magnetic, magnetic moments of adjacent atoms align in opposite direction so net magnetic moment is nil even in the presence of applied field.

In ferrimagnetic, magnetic moments of adjacent atoms are aligned opposite, but moments are not equal, so there is less magnetic moment. It is less than ferromagnetic.

In spite of weaker magnetic effects, some of ferri magnetic (Ferrites) have low electrical conductivity which are useful in cores of ac inductors & transformers.

A Super paramagnetic consists of ferro magnetic suspended in a dielectric binder. Particles contain many magnetic domains suspended in thin plastic tape. Tape stores large amount of information in magnetic form. They are used in audio, video and data recording systems.

Relative permeability for all materials:

Non magnetic : $\mu_r = 1$, vaccum.

Dia magnetic : $\mu_r \leq 1$, ex: Bismuth, silver, lead.

Para magnetic : $\mu_r \geq 1$, ex: Aluminium

Ferro magnetic : $\mu_r >> 1$, ex: Fe, ni, Co

Anti ferro magnetic : ex - MnO_2

Ferri magnetic : ex - Iron ferrite

Magnetization:

It is defined as net dipole moment per unit volume.

$$\text{Magnetization } M = \frac{m}{vol} = \frac{Q_m l}{vol} = \frac{Q_m l}{A l} = \frac{Q_m}{A}$$

where Q_m - pole strength of magnet

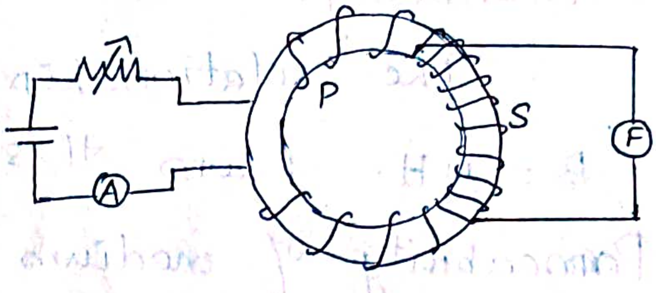
A - Area of cross section of bar magnet

l - axial length of bar magnet

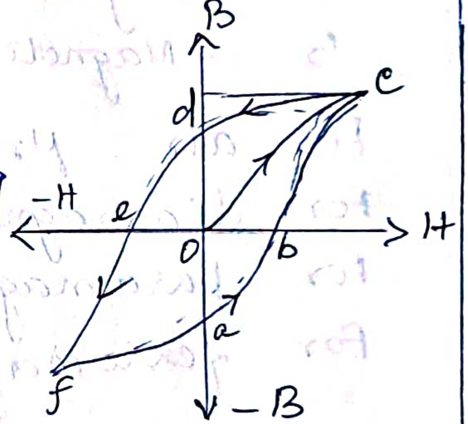
Magnetization Curve:

The ratio between B and H maybe constant for all values of H. The characteristic curve shows the variation of flux density (B) with field intensity (H) is called magnetization curve.

Consider a toroid with ferro magnetic core which has primary and secondary coils.



Primary coil is excited by variable DC supply which produces change in field intensity H & its corresponding effect measured by flux meter at secondary coil.



The value of H canbe increased or decreased by increasing or decreasing the current through the toroid.

H is increased from zero to a certain maximum value and corresponding value 'B' is noted. It reaches Saturation point 'c'.

If H is reduced to zero gradually, B willnot decrease to zero but has some value. This is residual or remanent value. To bring down the value of zero, H has to be applied in reverse direction. When H is reversed, B is reduced to zero. The field at H = -He is called coercive force.

H is further increased in negative direction, B reaches at negative saturation level. Again increases H, we get fabc curve in positive direction. This is called hysteresis. Resi. flux density on satu. loop is relectivity
coercive force on this " coercivity

In soft is easily magnetized materials, hyste loop is thin, with small area enclosed
hard " " area enclosed is greater

Permeability:

The relationship between B & H will be $B = \mu H$. From this,

Permeability of medium $\mu = B/H$.

$\mu = \mu_0 \mu_r$ H/m

$\mu_0 = 4\pi \times 10^{-7}$ H/m free space permeability.

μ_r = relative permeability.

H = Magnetic field intensity

B = Magnetic flux density.

For air, $\mu_r = 1$.

For diamagnetic, $\mu_r < 1$

For paramagnetic, $\mu_r > 1$

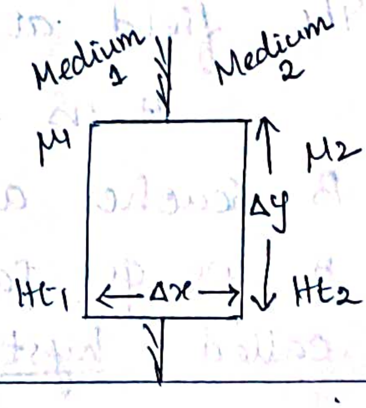
For ferromagnetic, μ_r = above 100

Magnetic Boundary Conditions:

1. The tangential component of magnetic field intensity is continuous across boundary.
2. The normal component of magnetic flux density is continuous across boundary.

Consider a boundary between two isotropic homogenius media with permeabilities μ_1 & μ_2.

Consider a small rectangle of width Δx & length Δy at boundary of two media. H_{t1} & H_{t2} are tangential components of medium 1 & 2.



According to Ampere's law, $\oint H \cdot dL = I$

If there is no current enclosed by the path

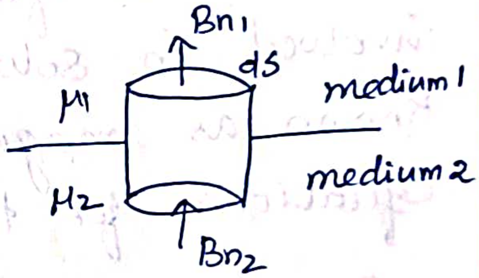
$$\oint H \cdot dl = 0.$$

$$H_{t_1} \Delta y - H_{t_2} \Delta y = 0$$

$\boxed{H_{t_1} = H_{t_2}}$ → Tangential component of H in medium 1 is same as 2. (ie) continuous

Consider a pill box of surface area ds across boundary between two isotropic media.

Let B_{n_1} & B_{n_2} are normal components of magnetic flux density in medium 1 & 2.



By Gauss law for magnetic field,

$$\iint_S B \cdot ds = 0.$$

$$B_{n_1} ds - B_{n_2} ds = 0$$

$\boxed{B_{n_1} = B_{n_2}}$ → The normal component of B is continuous across boundary.

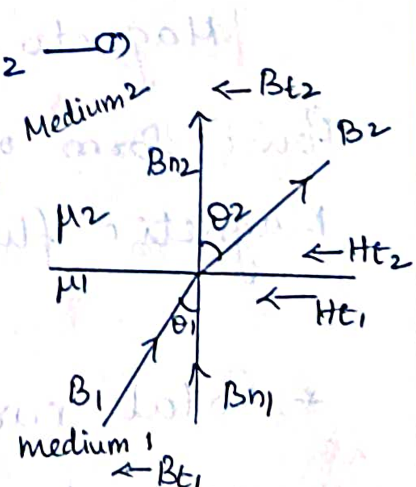
From the fig: $B_{n_1} = B_1 \cos\theta_1$, $B_{n_2} = B_2 \cos\theta_2$

But $B_{n_1} = B_{n_2}$ ∴ $B_1 \cos\theta_1 = B_2 \cos\theta_2$ —①

$H_{t_1} = B_{t_1}/\mu_1$; $H_{t_2} = B_{t_2}/\mu_2$

But $H_{t_1} = H_{t_2}$; $\dfrac{B_{t_1}}{\mu_1} = \dfrac{B_{t_2}}{\mu_2}$

$$\mu_2 B_{t_1} = \mu_1 B_{t_2}.$$



$B_{t_1} = B_1 \sin\theta_1$, $B_{t_2} = B_2 \sin\theta_2$

∴ $\mu_2 B_1 \sin\theta_1 = \mu_1 B_2 \sin\theta_2$ —②

Divide eqn (2) by (1), $\frac{\mu_2 B_1 \sin\theta_1}{B_1 \cos\theta_1} = \frac{\mu_1 B_2 \sin\theta_2}{B_2 \cos\theta_2}$

$$\mu_2 \tan\theta_1 = \mu_1 \tan\theta_2$$

$$\boxed{\frac{\tan\theta_1}{\tan\theta_2} = \frac{\mu_1}{\mu_2}}$$

The Magnetic Circuit:

The fundamental circuit techniques involved in solving a class of magnetic problems known as magnetic circuits. Now, write the equations for this circuit similar to electric circuit.

* First, relation between potential & electric field intensity is $\boxed{E = -\nabla V}$

Similarly, scalar magnetic potential with magnetic field intensity is $\boxed{H = -\nabla V_m}$

* Electric potential difference $V_{AB} = \int_A^B E \cdot dl$

Magneto motive force mmf $V_{mAB} = \int_A^B H \cdot dl$.

* Point form of ohm's law $J = \sigma E$

Magnetic flux density will be analog to current density
$B = \mu H$.

* Total current $I = \int_S J \cdot ds$

Magnetic flux $\phi = \int_S B \cdot ds$

* Resistance is ratio of mmf to potential difference to current. $V = IR$.

Reluctance is ratio of mmf to total flux.

(ie) $V_m = \phi \mathcal{R}$.

* Total resistance $R = \frac{d}{\sigma S}$ [d - length, σ - conductivity, S - Cross-section area]

Total reluctance $\mathcal{R} = \frac{d}{\mu S}$ [μ - permeability]

* In electric circuit, $\oint E \cdot dl = 0$

In magnetic circuit, $\oint H \cdot dl = I$

Potential energy and forces on magnetic materials

In electro static field, the general expression for energy is $W_E = \frac{1}{2} \int_v D \cdot E \, dv$.

The total energy stored in steady magnetic field in which B is related to H is

$$W_H = \frac{1}{2} \int_v B \cdot H \, dv$$

Let $B = \mu H$, then $W_H = \frac{1}{2} \int_v \mu H^2 dv$

$$= \frac{1}{2} \int_v \frac{B^2}{\mu} dv.$$

To calculate the forces on nonlinear magnetic materials, consider a long solenoid with silicon-steel core. A coil contains n turns/m with current I surrounds it.

Magnetic field intensity in core = nI A t/m.

Magnetic flux density is obtained from magnetization curve. Call this value as B_{st}. Suppose core is composed of two semi-infinite cylinders. Apply mechanical force to seperate these two sections of core while keeping flux density constant.

Apply force F over a distance dL, then work FdL. To determine the work in moving one core appears as stored energy in the air gap.

Increase in work $dW_H = FdL = \frac{1}{2}\frac{B_{st}^2}{\mu_0} S\,dl.$

where S - core cross-sectional area.

Thus, force $F = \frac{B_{st}^2 S}{2\mu_0}$

Inductance and Inductors:

Capacitors store electrical energy.
Inductors store magnetic energy.

Self Inductance:

By Faraday's law $\left[v = -\frac{d\phi}{dt}\right]$, changing current will produce an induced emf in the circuit to oppose the change in flux. This phenomena is self induction.

Self inductance of a circuit is the property of the circuit by which changing current induces emf in the circuit to oppose change in current.

Consider a coil having 'N' number of turns. If changing current is applied, emf is induced in the coil.

The induced emf is proportional to rate change of current $v \propto \frac{di}{dt}$. $\therefore v = L\frac{di}{dt}$

where L is self inductance.

From Faraday's law, the induced ~~of~~ emf, $v = N\frac{d\phi}{dt}$

$$\therefore L\frac{di}{dt} = N\frac{d\phi}{dt}$$

$$L = N\frac{d\phi}{di}$$

If permeability is constant, $\boxed{L = \frac{N\phi}{i}}$

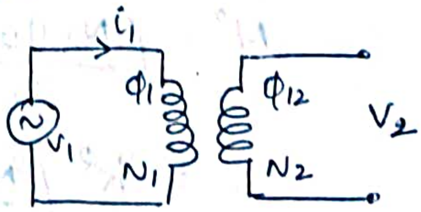
Inductance is ratio of total magnetic flux linkage to the current through the coil.

Mutual Inductance:

Consider two coils 1 & 2 magnetically coupled together. The changing current i_1 produces flux ϕ_1. If second coil is placed near first one, some of the flux links coil 2 say ϕ_{12}.

Induced emf in coil $V_2 = N_2\frac{d\phi_{12}}{dt}$

Since flux ϕ_{12} is produced by first coil current i_1, induced emf V_2 in coil 2 is proportional to the rate of change of current i_1.



$$V_2 \propto \frac{di_1}{dt}$$

$V_2 = M \frac{di_1}{dt}$; M - Mutual inductance between two coils.

$M \frac{di_1}{dt} = N_2 \frac{d\phi_{12}}{dt}$; $M = N_2 \frac{d\phi_{12}}{di_1}$

So, $M = \frac{N_2 \phi_{12}}{i_1}$.

Similarly, if flux ϕ_{21} is produced by second coil current i_2, induced emf V_1 in coil 1 is proportional to current i_2.

$V_1 \propto \frac{di_2}{dt}$, $V_1 = M \frac{di_2}{dt}$.

From Faraday's law, $V_1 = N_1 \frac{d\phi_{21}}{dt}$

$M = N_1 \frac{d\phi_{21}}{di_2}$; $\therefore M = N_1 \frac{\phi_{21}}{i_2}$.

Mutual inductance is ratio of induced magnetic flux linkage in one coil to the current through in other coil.

<u>Coefficient of coupling:</u>

The fraction of total flux produced by one coil linking a second coil is co-efficient of coupling (k).

$$k = \frac{\phi_{12}}{\phi_1} = \frac{\phi_{21}}{\phi_2}$$

Mutual inductance $M = \frac{N_2 \phi_{12}}{i_1}$ & $M = \frac{N_1 \phi_{21}}{i_2}$

$$M^2 = \left(\frac{N_2 \phi_{12}}{i_1}\right)\left(\frac{N_1 \phi_{21}}{i_2}\right) = \left(\frac{N_2 \phi_1 k}{i_1}\right)\left(\frac{N_1 \phi_2 k}{i_2}\right)$$

$$= k^2 \left(\frac{N_1 \phi_1}{i_1}\right)\left(\frac{N_2 \phi_2}{i_2}\right) = k^2 L_1 L_2$$

$$\therefore M = K\sqrt{L_1 L_2}$$

$$K = M / \sqrt{L_1 L_2}$$

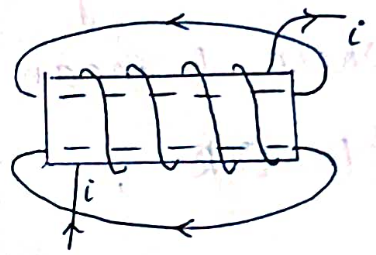
Consider two coupled coils of self inductances L_1 & L_2 and Mutual inductance M, then equivalent inductance $L = L_1 + L_2 + 2M$.

If both coils are connected in parallel,

equivalent inductance $L = \frac{L_1 L_2 - M^2}{L_1 + L_2 - 2M}$ (aiding) or $\frac{L_1 L_2 - M^2}{L_1 + L_2 + 2M}$ (Opposing)

Inductance of Solenoid:

Conductor which carries current produces magnetic field around it. If current in a coil changes, flux linkage also varies. So, value of flux dependson current in the coil.



Flux lines of a Solenoid.

Consider a Solenoid of N no. of turns carrying current I. If 'B' is flux density and A is area of cross-section of Solenoid, then

flux linkage $N\phi = NBA$.

For long solenoid, $B = \mu_0 \frac{NI}{l}$

$\therefore$ Inductance $L = \frac{N\phi}{I} = \frac{N \mu_0 N I A}{I l} = \mu_0 \frac{N^2 A}{l}$

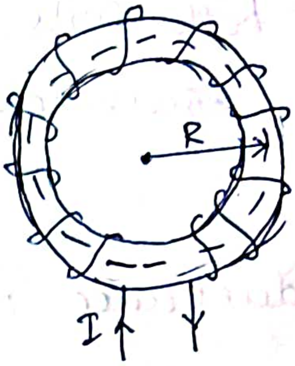
ex: Evaluate the inductance of solenoid of 2500 turns wound uniformly over a length of 0.5m on a cylindrical tube, 4 cm in diameter. Medium is air.

$$L = \mu_0 \frac{N^2 A}{l} = \frac{4\pi \times 10^{-7} \times 2500^2 \times \pi \times (0.04)^2 / 4}{0.5} = 19.7\ mH$$

Inductance of Toroid:

If long solenoid is bent into the form of ring & closed itself becomes toroid.

Consider a toroid of N no. of turns carrying current I with mean radius R.



Toroid.

Flux density $B = \mu_0 H = \mu_0 \frac{NI}{l}$.

where l - mean length of the coil.

$l = 2\pi R$, $\therefore B = \frac{\mu_0 NI}{2\pi R}$

Flux linkage in toroid is $N\phi = NBA = N\frac{\mu_0 NI}{2\pi R}.A$

Area of cross-section $A = \pi r^2$ [r - radius of coil]

$$\therefore N\phi = N.\frac{\mu_0 NI}{2\pi R}.\pi r^2 = \frac{\mu_0 N^2 r^2 I}{2R}$$

Inductance of Toroid is $L = \frac{N\phi}{I} = \frac{\mu_0 N^2 r^2}{2R}$

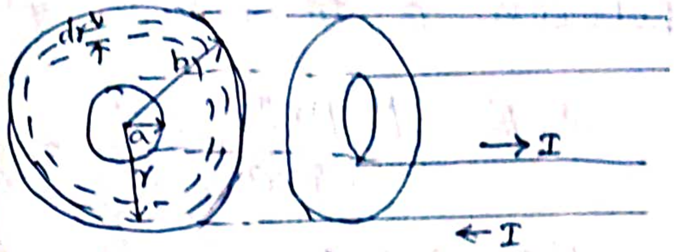
Ex: A toroidal coil of 1000 turns has mean radius of 20 cm and radius for winding 2 cm. What is average self inductance with air core & iron core of $\mu_r = 800$.

a) Air core : $\mu_r = 1$, $L = \frac{\mu_0 N^2 r^2}{2R} = \frac{4\pi \times 10^{-7} \times 1000^2 \times (0.02)^2}{2 \times 0.2}$

$$L = 1.257 \text{ mH}.$$

b) Iron Core: $\mu_r = 800$, $L = 800 \times 1.257 \text{ mH} = 1 \text{ H}$.

Inductance of coaxial Cable:



Consider a coaxial line with conducting cylinders of radius a & b. Consider metre length of line and radial thickness dr at a distance r from the centre of cable.

Flux passing thro' area (1×dr) is $B\,dr = d\phi$.

Flux density $B = \frac{\mu_0 I}{2\pi r}$

∴ flux $d\phi = \frac{\mu_0 I}{2\pi r}$

Total flux linkage per unit length between a & b.

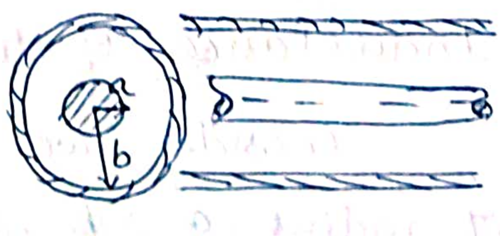
$$\phi = \int_a^b \frac{\mu_0 I}{2\pi r}\,dr = \frac{\mu_0 I}{2\pi} \ln r\Big]_a^b = \frac{\mu_0 I}{2\pi} \ln\frac{b}{a}.$$

∴ Inductance of a coaxial line per mtr length is

$$L = \frac{\phi}{I} = \frac{\mu_0}{2\pi} \ln\frac{b}{a} \text{ henry.}$$

$$L = 0.4606 \log_{10} b/a \text{ mH/km.}$$

Inductance of co axial cable with Solid inner Conductor:



Consider a co-axial cable with solid inner conductor of radius 'a' and outer radius 'b'. Let I be the current in solid conductor and −I be the current in outer conductor.

Flux density within solid conductor at a distance r from axis of cable is $B = \frac{\mu_0 \mu_r I' r}{2\pi a^2}$ $0 < r < a$.

The current flowing in solid inner conductor between a to r is $I' = \frac{I}{\pi a^2} \cdot \pi r^2 = \frac{I r^2}{a^2}$

$$\therefore B = \frac{\mu_0 \mu_r I}{2\pi a^2} \cdot \frac{I r^2}{a^2} = \frac{\mu_0 \mu_r I r^3}{2\pi a^4}.$$

The total flux linkage per unit length between o and a.

$$\phi = \int_0^a \frac{\mu_0 \mu_r I r^3}{2\pi a^4} \cdot dr$$

$$= \frac{\mu_0 \mu_r I}{2\pi a^4} \cdot \left.\frac{r^4}{4}\right]_0^a = \frac{\mu_0 \mu_r I a^4}{2\pi a^4 \cdot 4} = \frac{\mu_0 \mu_r I}{8\pi}$$

$\therefore$ Inductance $L_1 = \frac{\phi}{I} = \frac{\mu_0 \mu_r}{8\pi}$

Inductance of coaxial cable between a & b

$$L_2 = \frac{\mu_0}{2\pi} \ln \frac{b}{a}.$$

$\therefore$ Inductance of coaxial cable ~~between~~ per unit length

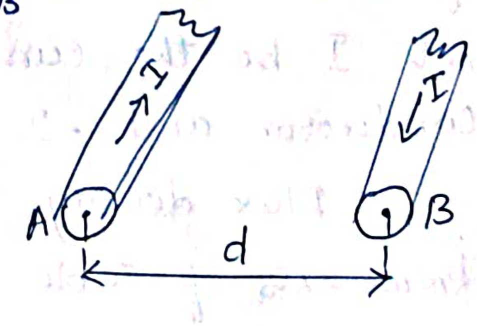
$$L = L_1 + L_2 = \frac{\mu_0 \mu_r}{8\pi} + \frac{\mu_0}{2\pi} \ln \frac{b}{a}.$$

$$L = \frac{\mu_0}{4\pi}\left[\frac{\mu_r}{2} + 2 \ln \frac{b}{a}\right] = 10^{-7}\left(\frac{\mu_r}{2} + 2 \ln \frac{b}{a}\right) \text{ H/m}$$

$$L = 10^{-7}\left[\frac{\mu_r}{2} + 4.606 \log_{10} \frac{b}{a}\right] \text{ mH/km}.$$

Inductance of two transmission Lines:

Consider two conductors A & B of radius a & b respectively and seperated by distance d. The conductor 'A' carries current of I & Conductor 'B' carries current of −I.



The internal flux linkage of conductor A is

$$\phi_1 = \frac{\mu_0 \mu_r I}{8\pi} \quad \text{(from previous)}$$

The external flux linkage with conductor A is

$$\phi_2 = \frac{\mu_0 I}{2\pi} \ln\left(\frac{d}{a}\right) \quad \text{(refer previous).}$$

Total flux linkage of A is $\phi = \phi_1 + \phi_2$.

$$\phi = \frac{\mu_0 \mu_r I}{8\pi} + \frac{\mu_0 I}{2\pi} \ln \frac{d}{a}.$$

Total inductance of conductor A is $L_A = \frac{\phi}{I}$.

$$L_A = \frac{\mu_0}{4\pi}\left[\frac{\mu_r}{2} + 2 \ln \frac{d}{a}\right] \text{ H/m}.$$

For conductor B, total flux linkage is

$$\phi = \frac{\mu_0 \mu_r I}{8\pi} + \frac{\mu_0 I}{2\pi} \ln \frac{d}{b}$$

Total inductance of conductor B is

$$L_B = \frac{\mu_0}{4\pi}\left[\frac{\mu_r}{2} + 2 \ln \frac{d}{b}\right]$$

Loop inductance of transmission line per unit length is

$$L = L_A + L_B = \frac{\mu_0}{4\pi}\left[\frac{\mu_r}{2} + 2\ln\frac{d}{a} + \frac{\mu_r}{2} + 2\ln\frac{d}{b}\right].$$

$$= \frac{\mu_0}{4\pi}\left[\mu_r + 2\ln\frac{d^2}{ab}\right]$$

$$= 10^{-7}\left[\mu_r + 4 \ln \frac{d}{\sqrt{ab}}\right] \text{ H/m}.$$

$$= 0.1\,\mu_r + 0.9212 \log_{10} \frac{d}{\sqrt{ab}} \text{ mH/km}.$$

Energy stored in magnetic field:

When a current through an inductor is increased from 0 to I with the potential difference across inductor is V, then energy supplied by source in time dt is given by, $dW = Vi\,dt$.

Energy stored in magnetic field is given by,

$$W = \int_0^I vi\,dt = \int_0^I L\frac{di}{dt}\cdot i\,dt = L\int_0^I i\,di.$$

$$W = L\frac{I^2}{2} = \frac{1}{2}LI^2.$$

Inductance of the solenoid is $L = \frac{\mu_0 N^2 A}{l}$.

Substitute value of L in above equation,

$$W = \frac{1}{2}\cdot\frac{\mu_0 N^2 A}{l}\cdot I^2 = \frac{1}{2}\mu_0\left(\frac{NI}{l}\right)^2 lA. \quad \left[\because \frac{NI}{l} = H\right]$$

$$W = \frac{1}{2}\mu_0 H^2 \cdot lA.$$

Energy stored per unit volume

$$w = \frac{1}{2}\mu_0 H^2 \ J/m^3. \quad [\because \text{volume} = lA]$$

$$= \frac{1}{2}(\mu_0 H)H.$$

Magnetic energy density

$$w = \frac{1}{2}BH \ J/m^3.$$

The energy stored in magneto-static field is

$$W = \int_v w\,dv = \frac{1}{2}\int_v B\cdot H\,dv = \frac{1}{2}\int \mu H^2\,dv.$$

Stokes' Theorem:

From Ampere's circuital law, we derive one of Maxwell's equation as

$$\nabla \times H = J \rightarrow \text{Point form of Ampere's circuital law.}$$

Consider surface S which is broken up into incremental surfaces of area Δs. If we apply the definition of curl to one of these incremental surfaces, then

$$\frac{\oint H \cdot dl_{\Delta s}}{\Delta s} = (\nabla \times H)_N .$$

where N indicates right hand normal to surface. The subscript $dl_{\Delta s}$ indicates that closed path is the perimeter of an incremental area Δs. This result may also be written as

$$\frac{\oint H \cdot dl_{\Delta s}}{\Delta s} = (\nabla \times H) \cdot a_N$$

(or) $$\oint H \cdot dl_{\Delta s} = (\Delta \times H \cdot a_N \Delta s) = (\nabla \times H) \cdot \Delta s$$

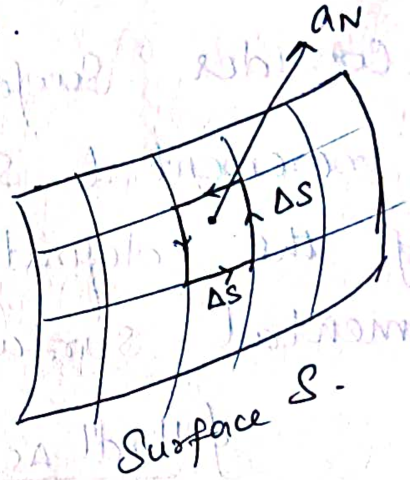
where a_N is a unit vector in the direction of the right-hand normal to Δs.

The circulation for every Δs comprising S and sum the results. To evaluate the closed line integral for each Δs, some cancellation will occur because every interior wall is covered once in each direction. The only boundaries on which cancellation can't occur from the outside boundary, the path enclosing S.

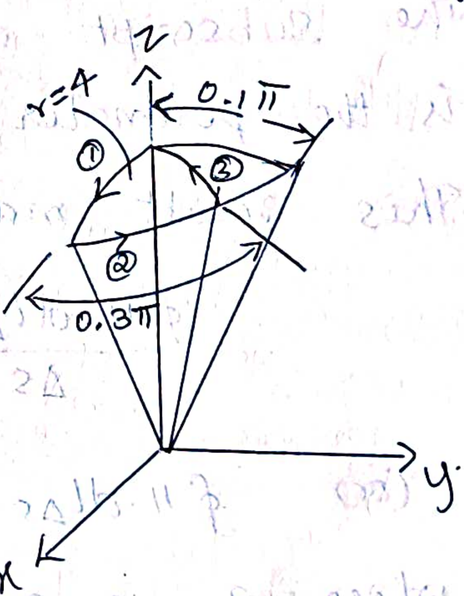
$$\therefore \oint H \cdot dl = \int_S (\nabla \times H) \cdot ds.$$

where dL is taken only on the perimeter S.

This is known as Stokes' theorem.



ex: 1) Consider the portion of sphere shown in fig. The surface is specified by $r = 4$, $0 \le \theta \le 0.1\pi$, $0 \le \phi \le 0.3\pi$ and the closed path forming its perimeter is composed of three circular arcs. We are given the field



$$H = 6r \sin\phi \, a_r + 18 r \sin\theta \cos\phi \, a_\phi$$

and are asked to evaluate each side of Stokes' theorem.

<u>Solution:</u> The first path is described in spherical coordinates by $r = 4$, $0 \le \theta \le 0.1\pi$, $\phi = 0$;

The second one by $r=4$, $\theta=0.1\pi$, $0\le\phi\le0.3\pi$ and the third by $r=4$, $0\le\theta\le0.1\pi$, $\phi=0.3\pi$

The differential path element dL is vector Sum of three differential lengths of spherical co-ordinate system.

$$dL = dr\, a_r + r d\theta\, a_\theta + r\sin\theta\, d\phi\, a_\phi .$$

The first term is zero on all three segments of path since $r=4$ and $dr=0$.

Second one is zero on segment 2 since θ is constant.

Third one is zero on both segments 1 & 3.

Thus,

$$\oint H.dl = \int_1 H_\theta r d\theta + \int_2 H_\phi r\sin\theta\, d\phi + \int_3 H_\theta r d\theta .$$

Since $H_\theta=0$, we have second integral to evaluate

$$\oint H.dl = \int_0^{0.3\pi} [18(4) \sin 0.1\pi \cos\phi]\, 4 \sin 0.1\pi\, d\phi .$$

$$= 288 \sin^2 0.1\pi \sin 0.3\pi = 22.2\,A$$

For Spherical coordinates,

$$\nabla\times H = \frac{1}{r\sin\theta}\left(\frac{\partial(H_\phi \sin\theta)}{\partial\theta} - \frac{\partial H_\theta}{\partial\phi}\right)a_r + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial H_r}{\partial\phi} - \frac{\partial(rH_\phi)}{\partial r}\right)a_\theta + \frac{1}{r}\left(\frac{\partial(rH_\theta)}{\partial r} - \frac{\partial H_r}{\partial\theta}\right)a_\phi .$$

$$\nabla \times H = \frac{1}{r\sin\theta}(36 r \sin\theta \cos\theta \cos\phi)\, a_r + \frac{1}{r}\left(\frac{1}{\sin\theta} 6r\cos\phi - 36 r \sin\theta \cos\phi\right) a_\theta$$

Since $ds = r^2 \sin\theta \, d\theta \, d\phi \, a_r$, the integral is

$$\int_s (\nabla \times H)\cdot ds = \int_0^{0.3\pi}\int_0^{0.1\pi} (36\cos\theta \cos\phi)\, 16 \sin\theta \, d\theta \, d\phi$$

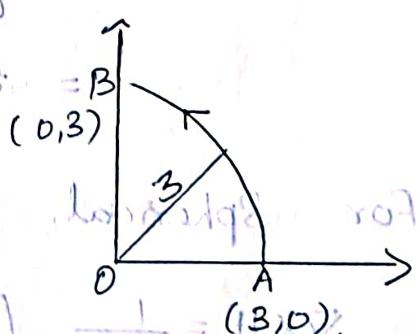
$$\int_0^{0.3\pi} 576\left(\tfrac{1}{2}\sin^2\theta\right)\Big|_0^{0.1\pi} \cos\phi \, d\phi$$

$$= 288 \sin^2 0.1\pi \cdot \sin 0.3\pi = 22.2\ A$$

$\oint H \cdot dl = \int_s (\nabla \times H)\, ds$, ∴ Stokes' theorem verified.

2) Given $\bar{F} = \hat{a}_x xy - \hat{a}_y 2x$, verify Stokes' theorem over a quarter circular disk with radius 3 in the first quadrant as shown in fig.

$$\oint F \cdot dl = \iint_s (\nabla \times F)\cdot \hat{n}\, ds$$



LHS

$$\oint_A^B F\cdot dl = \int_A xy\, dx - \int 2x\, dy$$

Circle equation $x^2 + y^2 = 9$

$\therefore \quad x = \sqrt{9 - y^2}$

$y = \sqrt{9 - x^2}$

UNIT IV. TIME-VARYING FIELDS AND MAXWELL'S EQUATIONS.

Faraday's Law:

Faraday proved that static magnetic field can't produce any current flow. But with a time varying field, an electro motive force induces which may drive a current in the closed path. This emf is voltage that induces from changing magnetic field or motion of conductors in a magnetic field.

Statement: "The electro motive force induced in a closed path is proportional to the rate of change of magnetic flux enclosed by the closed path". It can be stated as

$$e = -N\frac{d\phi}{dt} \text{ volts.}$$

N - No. of turns
e - induced emf.

Electro magnetic induction:

When a closed path moves in a magnetic field, current is generated and thus emf is generated. Similarly, when a closed path is kept steady and the magnetic field is varied, current is produced hence emf is generated. This phenomenon is electro magnetic induction.

Displacement Current & Maxwell Ampere law:

$$\nabla \times \bar{H} = J_c + \frac{\partial D}{\partial t}$$

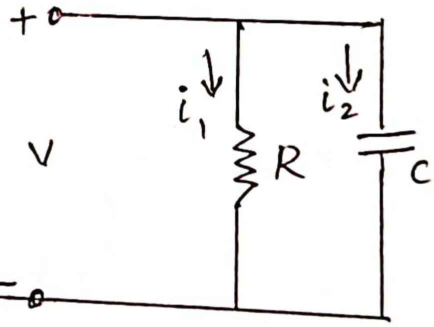
→ Here, first term J_c represents conduction current density

→ Second term represents current density in A/m^2. This quantity is obtained from time varying flux density. This is also called displacement current density (J_D).

$$\therefore \quad \nabla \times \bar{H} = \bar{J_c} + \bar{J_D}.$$

* Consider a parallel circuit of a resistor and capacitor driven by time varying voltage 'V'.

→ Current flowing thro' resistor is i_1 and it is conduction current because current is flowing due to actual motion of charges.



$$i_1 = V/R = i_c$$

* Conduction current density is given by,

$$\bar{J_c} = \frac{i_c}{A} = \sigma \bar{E}$$

* The current through the capacitor is the displacement current given by $i_2 = c\frac{dv}{dt}$.

* For static electromagnetic fields, Ampere's circuital law is $\nabla \times \bar{H} = \bar{J}$ —— ①

* Taking divergence on both sides,

$$\nabla \cdot (\nabla \times \bar{H}) = \nabla \cdot \bar{J} \quad \text{—— ②}$$

* But, $\nabla \cdot (\nabla \times \bar{F}) = 0$ according to vector identity,

$$\nabla \cdot (\nabla \times \bar{H}) = \nabla \cdot \bar{J} = 0 \quad \text{—— ③}$$

* Continuity equation is given by,

$$\nabla \cdot \bar{J} = -\frac{\partial \rho_v}{\partial t} \quad \text{—— ④}$$

Equations ③ & ④ are not compatible, so we can add one term "N" in ①

$$\nabla \times \bar{H} = \bar{J} + \bar{N} \quad \text{—— ④b.}$$

Take divergence on both sides,

$$\nabla \cdot (\nabla \times \bar{H}) = \nabla \cdot \bar{J} + \nabla \cdot \bar{N}$$

As $\nabla \cdot J = -\frac{\delta \rho_v}{\delta t}$, $\nabla \cdot \bar{N} = +\frac{\delta \rho_v}{\delta t}$ —— ⑤

According to Gauss law, $\nabla \cdot \bar{D} = \rho_v$

$$\nabla \cdot \bar{N} = \frac{\partial (\nabla \cdot \bar{D})}{\partial t} = \nabla \cdot \frac{\partial \bar{D}}{\partial t} \quad \text{—— ⑥}$$

Comparing we can write $N = \frac{\partial \bar{D}}{\partial t}$ —— ⑦.

$$i_D = i_a = \frac{\epsilon A}{d} \frac{dv}{dt} = \frac{\epsilon A}{d} \frac{d(Ed)}{dt} = \epsilon A \frac{dE}{dt}$$

Displacement current density $J_D = \frac{i_D}{A}$

$$J_D = \frac{\epsilon A \, dE/dt}{A} = \epsilon \frac{dE}{dt} = \frac{d(\epsilon E)}{dt} = \frac{\partial D}{\partial t}.$$

Total current density $\bar{J} = \bar{J_c} + \bar{J_D}$

$$\bar{J} = \sigma \bar{E} + \frac{\partial \bar{D}}{\partial t}$$

$$= \sigma \bar{E} + j\omega \epsilon \bar{E}$$

$$\frac{|\bar{J_c}|}{|\bar{J_D}|} = \frac{\sigma}{\omega \epsilon}$$ where $\sigma \rightarrow$ conductivity

If $\sigma/\omega\epsilon >> 1$, Medium is conductor

$\sigma/\omega\epsilon << 1$, Medium is dielectric.

Modified form of Ampere's circuital law,

$$\nabla \times \bar{H} = \bar{J_c} + \bar{J_D}$$

$$\nabla \times \bar{H} = J_c + \frac{\partial D}{\partial t}.$$

Maxwell's Equations:

Maxwell derived four equations to describe the electro magnetic field. These equations based on fundamental laws of Gauss, Faraday and Ampere.

From Ampere's circuital Law:

Ampere's law: The line integral of magnetic field intensity H on any closed path is equal to current enclosed by that path.

$$\int H \cdot dL = I$$

* The total current involves both conduction current and displacement current. A current through resistive element is conduction current. Current thro' capacitive element is displacement current

Conduction current density:

Ohm's law $I_c = V/R$

But, $R = \frac{\rho l}{A} = \frac{l}{\sigma A}$ where ρ - resistivity
σ - conductivity.
A - Area of cross-section of conductor.

$$I_c = \frac{V\sigma A}{l}$$

$$= \frac{El\sigma A}{l} = \sigma E A \quad ; \quad \text{But, } V = El$$

$$I_c/A = \sigma E = J_c \quad [J_c = I_c/A]$$

Conduction current density $\boxed{J_c = \sigma E}$

Displacement current Density:

Current thro' capacitor $I_D = \frac{dQ}{dt}$; $Q = CV$

$$C = \frac{\epsilon A}{d} \quad ; \quad V = Ed$$

$$I_D = C\frac{dv}{dt} = \frac{\epsilon A}{d}\frac{dv}{dt} = \frac{\epsilon A \cdot d}{d}\frac{dE}{dt}$$

$$\frac{I_D}{A} = \epsilon\frac{dE}{dt} = \frac{\partial D}{\partial t}. \qquad [\because D = \epsilon E]$$

Ampere's law can be written as,

$$\oint H.dl = \iint_S (J_c + I_D)\, ds = \oint_S \cancel{H}(\sigma E + \frac{\partial D}{\partial t})\, ds$$

$$\oint H.dl = \iint_S (\sigma E + \frac{\partial D}{\partial t})\, ds$$

$$\oint H.dl = \iint_S (J + \frac{\partial D}{\partial t})\, ds \quad \text{——— (1)}$$

Integral form of maxwell eqn (1)

By applying stoke's theorem,

$$\oint H.dl = \iint_S \nabla \times H \, ds$$

$$\iint_S (\nabla \times H)\, ds = \iint_S \left(J + \frac{\partial D}{\partial t}\right) ds$$

$$\nabla \times H = J + \frac{\partial D}{\partial t}$$ — Differential form of maxwell's eqn ①

From Faraday's law:

Faraday's law: Electro magnetic force induced in a circuit is equal to rate of decrease of magnetic flux linkage of the circuit.

(ie) $V = -\frac{d\phi}{dt} = -\frac{d}{dt}\left(\iint_S B.ds\right)$

But $V = \oint E.dl$.

$$\therefore \oint E.dl = -\frac{d}{dt}\left(\iint_S \bar{B}.ds\right)$$

$$\oint E.dl = -\iint_S \frac{\partial B}{\partial t}.ds$$ — ② Maxwell' eqn in integral form

$$\oint E.dl = -\mu \iint_S \frac{\partial H}{\partial t}.ds$$

Apply stokes' theorem,

$$\oint E.dl = \iint_S \nabla \times E . ds$$

$$\iint_S \nabla \times E.\, ds = -\left(\iint_S \frac{dB}{dt}.ds\right)$$

$$\nabla \times E = -\frac{dB}{dt}$$ — Differential form of maxwell eqn ②

$$\nabla \times E = -\mu \frac{dH}{dt}$$

From Electric ~~Magnetic~~ Gauss law:

Gauss law: Electric flux emerging through any closed surface is equal to charge enclosed by the surface.

$$\iint \bar{D} \cdot ds = Q$$

$$\iint \bar{D} \cdot ds = \iiint_V \rho \, dv$$ — ③ Maxwell's eqn in integral form.

Apply Divergence theorem,

$$\iint \bar{D} \cdot ds = \iiint_V \nabla \cdot \bar{D} \, dv$$

$$\iint \bar{D} \cdot ds = \iiint_V \nabla \cdot \bar{D} \, dv = \iiint_V \rho \, dv$$

$$\nabla \cdot D = \rho$$ — Point form of maxwell's equation

From magnetic Gauss law:

Gauss law: The total magnetic flux through any closed surface is equal to zero.

$$\iint \bar{B} \cdot ds = 0$$ — ④ Maxwell's eqn in integral form.

By applying divergence theorem,

$$\iint \bar{B} \cdot ds = \iiint_V \nabla \cdot \bar{B} \, dv$$

$$\iiint \nabla \cdot \bar{B} \, dv = 0$$

$$\nabla \cdot \bar{B} = 0$$ — Differential form of Maxwell's eqn.

Maxwell's equations:

Integral form	Point form
1. $\oint H \cdot dl = \iint \left(\bar{J} + \frac{\partial \bar{D}}{\partial t}\right) ds$	$\nabla \times H = J + \partial D / \partial t$
2. $\oint E \cdot dl = -\iint \partial \bar{B}/\partial t \cdot ds$	$\nabla \times E = -\partial B / \partial t$
3. $\iint \bar{D} \cdot ds = \iiint_V \rho \, dv$	$\nabla \cdot D = \rho$
4. $\iint \bar{B} \cdot ds = 0$	$\nabla \cdot B = 0$

In free space, no charges enclosed, $\sigma = 0$, $\rho = 0$.

1. $\oint \bar{H} \cdot dl = \iint_s \frac{\partial D}{\partial t} \cdot ds$ $\qquad \nabla \times H = \frac{\partial D}{\partial t}$

2. $\oint \bar{E} \cdot dl = -\iint_s \frac{\partial B}{\partial t} \cdot ds$ $\qquad \nabla \times E = -\frac{\partial B}{\partial t}$

3. $\iint_s \bar{D} \cdot ds = 0$ $\qquad \nabla \cdot D = 0$

4. $\iint_s \bar{B} \cdot ds = 0$ $\qquad \nabla \cdot B = 0$

Time Varying field:

The electric & magnetic field may be assumed as sinusoidal varying quantities. Time varying field is in phasor quantity

$$E(x,t) = Re[E(x) e^{j\omega t}]$$

$$\frac{dE}{dt}(x,t) = Re[j\omega E(x) e^{j\omega t}]$$

$$\nabla \times H = \sigma E + j\omega \epsilon E = (\sigma + j\omega \epsilon) E$$

$|||^{ly}$, $\nabla \times E = -j\omega \mu H$

∴ Maxwell's equations in phasor form are

1. $\oint H \cdot dl = \iint_s (\sigma + j\omega \epsilon) \bar{E}\, ds$ $\qquad \nabla \times \bar{H} = (\sigma + j\omega \epsilon) E$

2. $\oint E \cdot dl = -\iint_s j\omega \mu \bar{H} \cdot ds$ $\qquad \nabla \times \bar{E} = -j\omega \mu H$

3. $\iint \bar{D} \cdot ds = \iiint_v \rho\, dv$ $\qquad \nabla \cdot \bar{D} = \rho$

4. $\iint \bar{B} \cdot ds = 0$ $\qquad \nabla \cdot \bar{B} = 0$

Potential Functions:

The time varying potentials is called as retarded potentials. Scalar electric potential in terms of static charge distribution as

$$V = \iiint_v \frac{\rho \, dv}{4\pi\epsilon r}$$

Vector magnetic potential found from current distribution which is constant with time.

$$A = \iiint_v \frac{\mu J \, dv}{4\pi r}$$

Differential equations satisfied by V,

$$\nabla^2 V = -\rho/\epsilon \quad ; \quad \nabla^2 v = -\mu J$$

Fundamental fields $E = -\nabla V$, $B = \nabla \times A$

By adding unknown term, $E = -\nabla V + N$

Take curl, $\nabla \times E = -0 + \nabla \times N$

$$\nabla \times N = -\frac{\partial B}{\partial t} \rightarrow \text{Point form of Faraday's law}$$

$$\nabla \times N = -\frac{\partial}{\partial t}(\nabla \times A)$$

Simple solution of N is $N = -\frac{\partial A}{\partial t}$

$$E = -\nabla V - \frac{\partial A}{\partial t}$$

Take maxwell's equations $\nabla \times H = J + \frac{\partial D}{\partial t}$

$$\nabla \cdot D = \rho$$

$$\frac{1}{\mu} \nabla \times \nabla \times A = J + \epsilon\left(-\nabla \frac{\partial V}{\partial t} - \frac{\partial^2 A}{\partial t^2}\right)$$

and $\epsilon\left[(-\nabla \cdot \nabla V) - \frac{\partial}{\partial t} \nabla \cdot A\right] = \rho$

(or) $\nabla(\nabla \cdot A) - \nabla^2 A = \mu J - \mu\epsilon\left(\nabla \frac{\partial V}{\partial t} + \frac{\partial^2 A}{\partial t^2}\right)$

and $\nabla^2 V + \frac{\partial}{\partial t}(\nabla \cdot A) = -\rho/\epsilon$

$\nabla \cdot A = -\mu\epsilon \frac{\partial V}{\partial t}$, velocity $v = \frac{1}{\sqrt{\mu\epsilon}}$.

$\nabla^2 A = -\mu J + \mu\epsilon \frac{\partial^2 A}{\partial t^2}$

$\nabla^2 V = -\rho/\epsilon + \mu\epsilon \frac{\partial^2 V}{\partial t^2}$

These equations are related to wave equations

here, $V = \int_v \frac{\rho_v \, dv}{4\pi\epsilon R}$

where ρ_v - every t appears in expression for ρ_v has been replaced by retarded time.

$T = t - R/v$

If charge density throughout space was

$\rho_v = e^{-r} \cos \omega t$

$[\rho_v] = e^{-r} \cos[\omega(t - R/v)]$

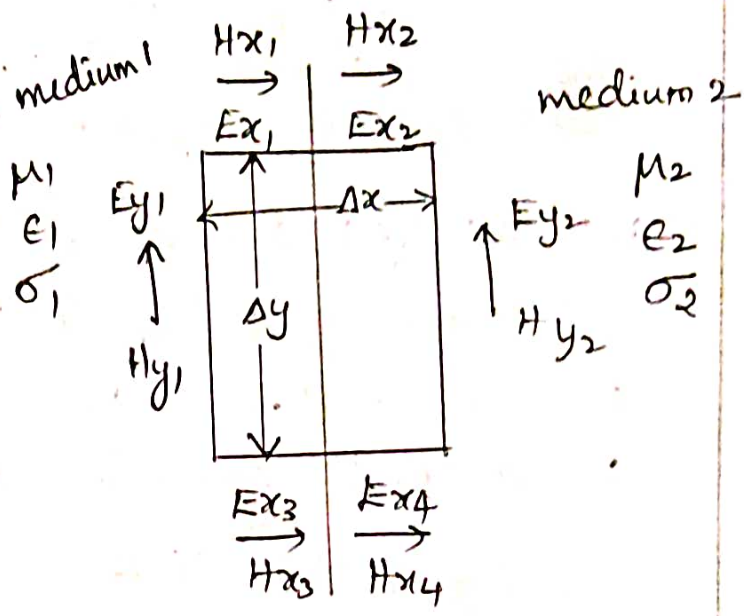
R - distance between differential element of charge being considered.

Retarded vector magnetic potential

$A = \int_v \frac{\mu J \, dv}{4\pi R}$

Electro Magnetic Boundary Conditions:

Consider a small rectangle with width Δx, length Δy at the boundary enclosing a small portion of each media.



Maxwell's equation

$$\int E \cdot dl = -\iint_s \frac{\partial B}{\partial t} \cdot ds$$

This is applied to rectangle,

$$Ey_1 \Delta y + Ex_1 \frac{\Delta x}{2} + Ex_2 \frac{\Delta x}{2} - Ey_2 \Delta y - Ex_4 \frac{\Delta x}{2} - Ex_3 \frac{\Delta x}{2} = -\frac{dB}{dt} \cdot \Delta x \Delta y$$

Consider the area of rectangle is made to approach to zero by reducing the width Δx to approach zero.

Then, $Ey_1 \Delta y - Ey_2 \Delta y = 0$

$$\boxed{Ey_1 = Ey_2}$$

Tangential component of E is continuous.

First Maxwell's equation is

$$\oint H \cdot dl = \iint_s \left(J + \frac{\partial D}{\partial t}\right) ds$$

Apply to rectangle,

$$Hy_1 \Delta y + Hx_1 \frac{\Delta x}{2} + Hx_2 \frac{\Delta x}{2} - Hy_2 \Delta y - Hx_4 \frac{\Delta x}{2} - Hx_3 \frac{\Delta x}{2} = \left(J + \frac{\partial D}{\partial t}\right) \Delta x \Delta y$$

If $\Delta x \to 0$, then $H_{y_1}\Delta y - H_{y_2}\Delta y = 0$

$$\boxed{H_{y_1} = H_{y_2}}$$

Tangential component of H is continuous.

For a perfect conductor, high frequency current will flow in a thin sheet near the surface. In a current sheet, linear current density J_1 flows in a sheet of depth Δx.

If $\Delta x \to 0$, $J \cdot \Delta x = J_1$ A/m.

If maxwell's eqn ① is applied to rectangle, then

$$H_{y_1}\Delta y + H_{x_1}\frac{\Delta x}{2} + H_{x_2}\frac{\Delta x}{2} - H_{y_2}\Delta y - H_{x_4}\frac{\Delta x}{2} - H_{x_3}\frac{\Delta x}{2} = \left(J + \frac{\partial D}{\partial t}\right)\Delta x\,\Delta y.$$

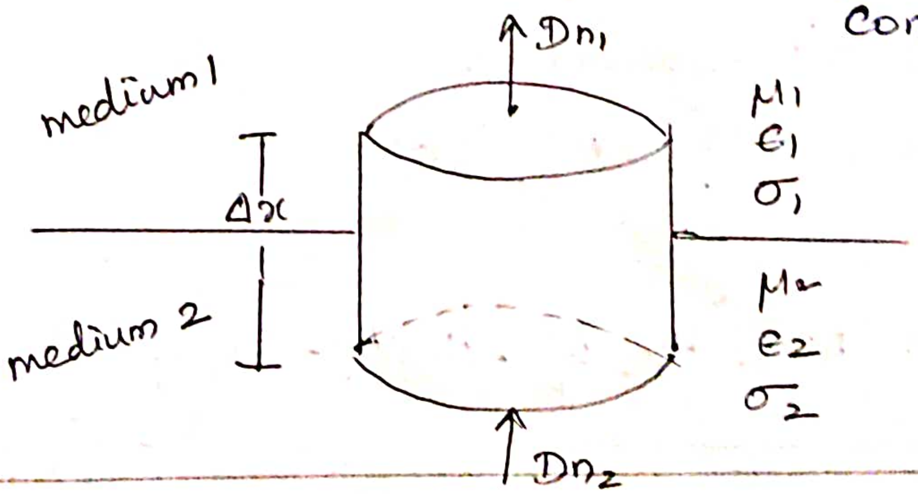
$$= J\,\Delta x\,\Delta y + \frac{\partial D}{\partial t}\Delta x\,\Delta y$$

$$= J_1\,\Delta y + \frac{\partial D}{\partial t}\Delta x\,\Delta y$$

If $\Delta x \to 0$, then $H_{y_1}\Delta y - H_{y_2}\Delta y = J_1\,\Delta y$

$$\boxed{H_{y_1} - H_{y_2} = J_1}$$

Tangential component of H is discontinuous by linear current density at surface of perfect conductor.



Consider a pill box of volume $ds.\Delta x$ at the boundary between two media.

ds - area of flat surfaces of pill box

Δx - their seperation

ρ - Volume charge density

Maxwell's third equation $\iint \bar{D} \cdot ds = \iiint_v \rho . dv$

Apply to pill box at boundary,

$$D_{n_1} ds - D_{n_2} ds = \rho ds . \Delta x$$

If $\Delta x \to 0$, then $D_{n_1} ds - D_{n_2} ds = 0$

$$\boxed{D_{n_1} = D_{n_2}}$$

Normal component of D is continuous if there is no surface charge density.

Maxwell's 3rd equation is applied to pill box.

$$D_{n_1} ds - D_{n_2} ds = \rho dx$$

If $\Delta x \to 0$, then. $\boxed{D_{n_1} - D_{n_2} = \rho}$

Normal component of D is discontinuous if there is surface charge density

Maxwell's fourth equation is

$$\iint \bar{B} \cdot ds = 0$$

Apply to pill box at boundary $B_{n_1} ds - B_{n_2} ds = 0$

$$\boxed{B_{n_1} = B_{n_2}}$$

Normal component of B is continuous across boundary

Wave Equations and Solutions:

Electro magnetic wave equation can be obtained from Maxwell's equations.

Maxwell's equation from Faraday's law in point form as

$$\nabla \times E = -\frac{\partial B}{\partial t} = -\mu \frac{\partial H}{\partial t}$$

Take Curl, $\nabla \times \nabla \times E = -\mu \, \nabla \times \frac{\partial H}{\partial t}$ —— ①

Maxwell's eqn from Ampere's law in point form as

$$\nabla \times H = J + \frac{\partial D}{\partial t} = \sigma E + \epsilon \frac{\partial E}{\partial t}$$

Differentiating, $\nabla \times \frac{\partial H}{\partial t} = \frac{d}{dt}\left(\sigma E + \epsilon \frac{\partial E}{\partial t}\right)$

$$\nabla \times \frac{\partial H}{\partial t} = \sigma \frac{\partial E}{\partial t} + \epsilon \frac{\partial^2 E}{\partial t^2} \quad \text{—— ②}$$

Substitute ② in eqn ①

$$\nabla \times \nabla \times E = -\mu \left[\sigma \frac{\partial E}{\partial t} + \epsilon \frac{\partial^2 E}{\partial t^2}\right]$$

$$\nabla \times \nabla \times E = -\mu\sigma \frac{\partial E}{\partial t} - \mu\epsilon \frac{\partial^2 E}{\partial t^2} \quad \text{—— ③}$$

According to identity,

$$\nabla \times \nabla \times E = \nabla(\nabla . E) - \nabla^2 E \quad \text{—— ④}$$

But $\nabla . E = \frac{\nabla . D}{\epsilon}$; There is no net charge within conductor $\rho = 0$.

$\nabla . E = \nabla . D = 0$

Eqn ④ becomes $\nabla \times \nabla \times E = -\nabla^2 E$ —— ⑤

Compare (3) & (5),

$$\nabla^2 E = -\mu\sigma \frac{\partial E}{\partial t} - \mu\epsilon \frac{\partial^2 E}{\partial t^2} = 0 \quad \text{—— (6)}$$

This is wave equation for electric field E.

Similarly, for magnetic field,

Maxwell's equation $\nabla \times H = \sigma E + \epsilon \frac{\partial E}{\partial t}$

Take curl, $\nabla \times \nabla \times H = \sigma \nabla \times E + \epsilon \nabla \times \frac{\partial E}{\partial t}$ —— (7)

Maxwell's equation $\nabla \times E = -\mu \frac{\partial H}{\partial t}$

Differentiating, $\nabla \times \frac{\partial E}{\partial t} = -\mu \frac{\partial^2 H}{\partial t^2}$.

Substitute, $\nabla \times E$ & $\nabla \times \frac{\partial E}{\partial t}$ in eqn (7)

$$\nabla \times \nabla \times H = -\mu\sigma \frac{\partial H}{\partial t} - \mu\epsilon \frac{\partial^2 H}{\partial t^2} \quad \text{—— (8)}$$

But identity, $\nabla \times \nabla \times H = \nabla(\nabla \cdot H) - \nabla^2 H$

But $\nabla \cdot \bar{B} = \nabla \cdot \bar{H} = 0$, then $\nabla \times \nabla \times H = -\nabla^2 H$ —— (9)

Compare eqns (8) & (9),

$$\nabla^2 H - \mu\sigma \frac{\partial H}{\partial t} = \mu\epsilon \frac{\partial^2 H}{\partial t^2}$$

$$\nabla^2 H - \mu\sigma \frac{\partial H}{\partial t} - \mu\epsilon \frac{\partial^2 H}{\partial t^2} = 0 \quad \text{—— (10)}$$

This is Wave equation for magnetic field.

For free space, $\sigma = 0$, $\rho = 0$

Maxwell's equations $\nabla \times E = -\frac{\partial B}{\partial t} = -\mu \frac{\partial H}{\partial t}$

Take curl, $\nabla \times \nabla \times E = -\mu \nabla \times \frac{\partial H}{\partial t}$

But, $\nabla \times H = \frac{\partial D}{\partial t} = \epsilon \frac{\partial E}{\partial t}$

$$\nabla \times \frac{\partial H}{\partial t} = \frac{\partial}{\partial t}(\nabla \times H) = \frac{\partial}{\partial t}\left(\epsilon \frac{\partial E}{\partial t}\right) = \epsilon \frac{\partial^2 E}{\partial t^2}$$

$$\therefore \nabla \times \nabla \times E = -\mu \epsilon \frac{\partial^2 E}{\partial t^2}$$

But, $\nabla \times \nabla \times E = \nabla(\nabla \cdot E) - \nabla^2 E$

$\nabla \cdot E = \frac{1}{\epsilon} \nabla \cdot D = \rho/\epsilon = 0$, then $\nabla \times \nabla \times E = -\nabla^2 E$

$$\therefore \boxed{\nabla^2 E = \mu \epsilon \frac{\partial^2 E}{\partial t^2}}$$ $\rightarrow$ wave equation for free space.

Similarly, for magnetic field,

$$\boxed{\nabla^2 H = \mu \epsilon \frac{\partial^2 H}{\partial t^2}}$$

Time Harmonic Fields:

The time dependence of the field quantities depends on the source functions. One of the most important case of time varying EM field is the time harmonic (sine (or) cosine) time variation where excitation of source varies sinusoidally in time with a single frequency.

For time-harmonic fields, phasor analysis can be applied to obtain single frequency steady state response. Since, Maxwell's equations are linear differential equations, for source functions with arbitrary time dependence, electromagnetic fields can be determined by superposition. Periodic time functions can be expanded into Fourier series of harmonic sinusoidal components while transient non-periodic functions can be expressed as Fourier integrals.

Field vectors that vary with space coordinates and are sinusoidal function of time can be represented in terms of vector phasors that depend on the space coordinates.

$$E(x, y, z, t) = Re\left[E(x, y, z)\, e^{j\omega t}\right].$$

where $E(x, y, z)$ is a vector phasor that contain the information on direction, magnitude and phase.

Problems:

1) In a cylindrical conductor to the region $0.01 \le r \le 0.02$, $0 < z < 1$ m and the current density is given by, $\bar{J} = 10 e^{-100r} \hat{a}_\phi$ A/m^2 Find the total current crossing the extensial of this region with ϕ = constant plane.

Sol: Total current in the wire is given as,

$$I = \int_s J \cdot ds = \int_{r=0.01}^{0.02} \int_{z=0}^{1} [10 e^{-100r} \hat{a}_\phi][r dr dz \hat{a}_\phi]$$

$$= \int_{r=0.01}^{0.02} \int_{z=0}^{1} 10 r e^{-100r} dr dz$$

$$= 10 \int_{r=0.01}^{0.02} r e^{-100r} dr$$

$$= 10 \left[\frac{r e^{-100r}}{-100} \Big|_{0.01}^{0.02} - \int_{r=0.01}^{0.02} \frac{e^{-100r}}{-100} dr \right]$$

$$= 10 \left[-\frac{1}{100}(0.02 e^{-2} - 0.01 e^{-1}) + \frac{e^{-100r}}{-100 \times 100} \Big|_{0.01}^{0.02} \right]$$

$$= 2 \times 10^{-3} e^{-1} - 3 \times 10^{-3} e^{-2}$$

$$= 0.033 \text{ mA}$$

2) Find the total current in a circular conductor of radius 4 mm if the current density varies according to $J = 10^4 / r$ A/m^2.

Total current $I = \int_s J \cdot ds = \int_{\phi=0}^{2\pi} \int_{r=0}^{0.004} \frac{10^4}{r} r dr d\phi$

$$I = 2\pi \times 10^4 \int_{r=0}^{0.004} dr = 2\pi \times 10^4 \times 0.004 = 80\pi \ A$$

3) In a certain region, current density vector is given by, $J = 3x\,\hat{a_x} + (y-3)\,\hat{a_y} + (2+z)\,\hat{a_z}$ A/m^2. Find the total current flowing out of the surface of the box bounded by five planes $x=0$, $y=0$, $z=0$ and $(3x+z)=3$, $y=2$.

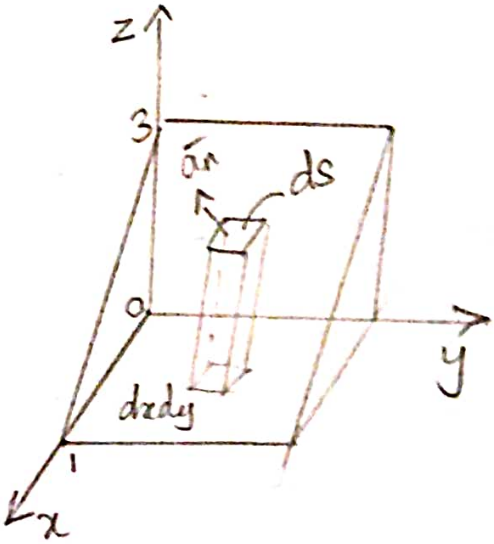
Solu. Normal vector to be always out of box so $\int_s J.ds$ gives the current flows out of surface.

→ For surface $x=0$,

$$\bar{J} = (y-3)\bar{a_y} + (2+z)\bar{a_z}$$

$$ds = -dy\,dz\,\hat{a_x}$$

$$\therefore I = \int J.ds = 0.$$



→ For surface $y=0$,

$$\bar{J} = 3x\bar{a_x} - 3\bar{a_y} + (2+z)\bar{a_z}$$

$$ds = -dx\,dz\,\bar{a_y}$$

$$\therefore I = \int_s J.ds = \int_{x=0}^{1}\int_{z=0}^{3-3x} 3\,dz\,dx$$

$$= \int_{x=0}^{1} 3(3-3x)dx = \left[9x - \frac{9}{2}x^2\right]_0^1 = \frac{9}{2} A$$

→ For surface $y=2$,

$$\bar{J} = 3x\bar{a_x} - \bar{a_y} + (2+z)\bar{a_z}$$

$$ds = dx\,dy\,\bar{a_y}.$$

$$\therefore \quad I = \int_S J.ds = \int_{x=0}^{1}\int_{z=0}^{3-3x} -dz\,dx = \int_{x=0}^{1} (-3-3x)\,dx$$

$$= \left[-3x + \frac{3}{2}x^2\right]_0^1 = -\frac{3}{2}\,A$$

→ For surface $z=0$,

$$\bar{J} = 3x\,\bar{a}_x + (y-3)\,\bar{a}_y + 2\bar{a}_z$$

$$d\bar{s} = -dx\,dy\,\bar{a}_z$$

$$\therefore \quad I = \int_S J.ds = \int_{y=0}^{2}\int_{x=0}^{1} -2\,dx\,dy = -4\,A$$

→ For surface $(3x+z) = 3$,

$$\bar{J} = 3x\,\bar{a}_x + (y-3)\,\bar{a}_y + (5-3x)\,\bar{a}_z$$

$$d\bar{s} = (3\bar{a}_x + 1\bar{a}_z)\,dx\,dy$$

$$\therefore \quad I = \int_S J.ds = \int_{y=0}^{2}\int_{x=0}^{1} (9x+5-3x)\,dx\,dy$$

$$= \int_{x=0}^{1} (18x+10-6x)\,dx = \int_0^1 (12x+10)\,dx = 16\,A$$

Adding the components of the currents, total current flowing out of the closed surface is

$$I = \left(0 + \frac{9}{2} - \frac{3}{2} - 4 + 16\right) = 15\,A.$$

4) If $\bar{J} = \frac{1}{r^3}(2\cos\theta\,\hat{a}_r + \sin\theta\,\hat{a}_\theta)\ A/m^2$, Calculate current passing through a) A hemispherical shell of radius 20cm
b) A spherical shell of radius 10 cm.

Soln: Total current is given as, $I = \int_S J.ds$

Here, $ds = r^2\sin\theta\,d\phi\,d\theta\,\hat{a}_r$

a) Total current passing thro' hemispherical shell of radius 20 cm is

$$I = \int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi} \frac{1}{r^3}(2\cos\theta\,\hat{a}_r + \sin\theta\,\hat{a}_\theta)\cdot(r^2\sin\theta\, d\phi\, d\theta\,\hat{a}_r)\Big|_{r=0.2}$$

$$= \int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi} \frac{1}{r^3}\, 2\cos\theta\, r^2 \sin\theta\, d\phi\, d\theta\Big|_{r=0.2}$$

$$= \pi \times \frac{2}{r}\int_{\theta=0}^{\pi/2} \sin\theta\, d(\sin\theta)\Big|_{r=0.2}$$

$$= \frac{4\pi}{0.2}\left[\frac{\sin^2\theta}{2}\right]_0^{\pi/2} = 10\pi = 31.42\text{ A}$$

b) Total current passing thro' spherical shell of radius 10 cm is

$$I = \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \frac{1}{r^3}(2\cos\theta\,\hat{a}_r + \sin\theta\,\hat{a}_\theta)\cdot(r^2\sin\theta\, d\phi\, d\theta\,\hat{a}_r)\Big|_{r=0.1}$$

$$= \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \frac{1}{r^3}\, 2\cos\theta\, r^2\sin\theta\, d\phi\, d\theta\Big|_{r=0.1}$$

$$= 2\pi \times \frac{2}{r}\int_{\theta=0}^{\pi} \sin\theta\, d(\sin\theta)\Big|_{r=0.1}$$

$$= \frac{4\pi}{0.1}\left[\frac{\sin^2\theta}{2}\right]_0^{\pi} = 0$$

5) For the current density $\bar{J} = 10z\sin^2\phi\,\hat{a}_r$ A/m^2, Find the current thro' the cylindrical surface $r = 2$, $1 \le z \le 5$ m.

Total current passing thro' cylindrical surface is

$$I = \int_S \bar{J}\cdot d\bar{s} = \int_{z=1}^{5}\int_{\phi=0}^{2\pi} (10z\sin^2\phi\,\hat{a}_r)\cdot(r\, d\phi\, dz\,\hat{a}_r)\Big|_{r=2}$$

$$= 10r\left[\frac{z^2}{2}\right]_1^5 \int_0^{2\pi} \sin^2\phi\, d\phi \Big]_{r=2}$$

$$= 10 \times 2 \times \frac{24}{2} \times \frac{2\pi}{2} = 240\pi = 754\ A.$$

6) Find the current in a circular wire of radius 2mm when the current density in the conductor is $J = 30(1-e^{-1000r})\hat{a_z}\ A/m^2$.

Soln: Total current in the wire is

$$I = \int_S J.ds = \int_{\phi=0}^{2\pi}\int_{r=0}^{0.002} 30(1-e^{-1000r})\hat{a_z} \cdot r\,dr\,d\phi\ \hat{a_z}$$

$$= \int_{\phi=0}^{2\pi}\int_{r=0}^{0.002} 30(1-e^{-1000r})\, r\,dr\,d\phi$$

$$= 60\pi \int_{r=0}^{0.002} (1-e^{-1000r})\, r\,dr$$

$$= 60\pi\left[\frac{r^2}{2}\right]_0^{0.002} - 60\pi\left[\frac{re^{-1000r}}{-1000}\Big|_0^{0.002} - \int_0^{0.002}\frac{e^{-1000r}}{-1000}dr\right]$$

$$= 60\pi \times 2\times10^{-6} - 60\pi\left[-2\times10^{-6}\times e^{-2} + \frac{e^{-1000r}}{-1000\times1000}\Big|_0^{0.002}\right]$$

$$= 120\pi\times10^{-6} + 120\pi\times10^{-6}\times e^{-2} + 60\pi\times10^{-6}\times e^{-2} + (-60\pi\times10^{-6})$$

$$= 60\pi\times10^{-6}(1+3e^{-2})$$

$$= 265.03\ \mu A.$$

7) The free charge density of Cu is $1.81\times10^{10}\ C/m^3$. For a current density of $8\times10^6\ A/m^2$, find the electric field intensity and drift velocity. Take conductivity of Cu is $5.8\times10^7\ \mho/m$.

Soln: $\rho = 1.81\times10^{10}\ C/m^3$, $J = 8\times10^6\ A/m^2$, $\sigma = 5.8\times10^7$

Electric field intensity is $E = \frac{J}{\sigma}$

$$E = \frac{8\times10^6}{5.8\times10^7} = 0.138\ V/m.$$

Drift velocity $v = \frac{J}{\rho} = \frac{8\times10^6}{1.81\times10^{10}} = 4.42\times10^{-4}\ m/s$

8) Find J_c and J_d for moist soil which has conductivity $\sigma = 10^{-3}\ \mho/m$, $\epsilon_r = 2.5$;

Given $E = 6\times10^{-6}\sin(9\times10^9 t)\ V/m$.

Soln: conduction current density is

$$\bar{J_c} = \sigma\bar{E} = 10^{-3}\times6\times10^{-6}\sin(9\times10^9 t)$$
$$= 6\times10^{-9}\sin(9\times10^9 t)\ A/m^2$$

Displacement current density

$$\bar{J_d} = \epsilon\frac{\partial E}{\partial t} = \epsilon_0\epsilon_r\frac{\partial}{\partial t}[6\times10^{-6}\sin 9\times10^9 t]$$
$$= 2.5\times8.854\times10^{-12}\times6\times10^{-6}\times10^9\cos(9\times10^9 t)$$
$$= 1.19\times10^{-6}\cos(9\times10^9 t)\ A/m^2.$$

9) S.T displacement thro' capacitor is conduction current if the supply volt is $v = V_m\sin\omega t$.

Soln: Applied Volt $v = V_m\sin\omega t$; $\quad C = \frac{\epsilon A}{d}$.

Conduction current $I_d = C\frac{dv}{dt} = \frac{\epsilon A}{d}\frac{d}{dt}[V_m\sin\omega t]$

$$= V_m\omega\frac{\epsilon A}{d}\cos\omega t \quad \text{①}$$

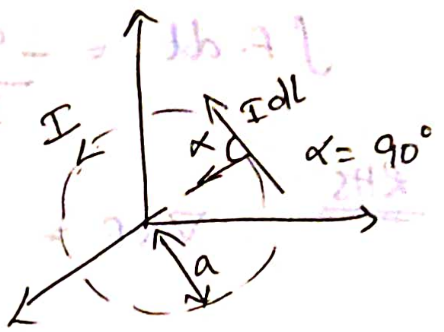
Displacement current $I_d = \epsilon A\frac{dE}{dt} = \epsilon A\frac{d}{dt}\left[\frac{v}{d}\right]$

$$= \frac{\epsilon A}{d}\frac{d}{dt}(V_m\sin\omega t) = V_m\omega\frac{\epsilon A}{d}\cos\omega t \quad \text{②}$$

① = ② $\therefore$ displacement current = Conduction current

Problems:

1) Find an expression for H at the centre of circular wire carrying current I in anticlockwise direction. Radius of circle is 'a' and wire is in xy plane.



Field intensity $H = \oint dH$

dH - Field intensity due to current element Idl.

Direction of dl at P is given by tangent at point P. Unit vector directed towards center along radius. $\therefore \alpha = 90°$.

$$|dH| = \frac{Idl}{4\pi a^2} \sin 90° = \frac{Idl}{4\pi a^2}.$$

Direction of vector dH is $dl \times u_r$ along z axis in positive direction.

$$dH = \frac{Idl}{4\pi a^2} \hat{z}.$$

$$\therefore H = \hat{z} \int \frac{Idl}{4\pi a^2} = \hat{z} \frac{I}{4\pi a^2} \cdot l = \frac{I}{4\pi a^2} \cdot 2\pi a \; \hat{z}$$

$$= \frac{I}{2a} \hat{z}$$

$$\boxed{H = \frac{I}{2a} \hat{z}}$$

$$\therefore \int F\cdot dl = \int_3^0 x\sqrt{9-x^2}\,dx - 2\int_0^3 \sqrt{9-y^2}\,dy$$

$$= -\tfrac{1}{3}(9-x^2)^{3/2}\Big|_3^0 - \left[y\sqrt{9-y^2} + 9\sin^{-1} y/3\right]_0^3$$

where $\int \sqrt{a^2-x^2}\,dx = \left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1} x/a\right]$.

$$\int F\cdot dl = -\frac{9^{3/2}}{3} - 9\,\pi/2 = -9(1+\pi/2) \quad \text{—— (1)}$$

<u>RHS</u>

$$\nabla\times F = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy & -2x & 0 \end{vmatrix}$$

$$= \hat{a}_z(-2-x) = -\hat{a}_z(x+2)$$

$$\iint_S (\nabla\times F)\,ds = \int_0^3\int_0^{\sqrt{9-y^2}} (\nabla\times F)\,a_z\,dx\,dy.$$

$$= \int_0^3\int_0^{\sqrt{9-y^2}} -(x+2)\,dx\,dy$$

$$= -\int_0^3 \left[\frac{x^2}{2}+2x\right]_0^{\sqrt{9-y^2}} dy.$$

$$= -\int_0^3 \frac{9-y^2}{2} + 2\sqrt{9-y^2}\,)\,dy$$

$$= -\left[9/2\,y - y^3/6 + y\sqrt{9-y^2} + 9\sin^{-1} y/3\right]_0^3$$

$$= -9(\pi/2+1) \quad \text{—— (2)}$$

(1) = (2), Stokes' theorem verified.

2) A circular coil of radius 10 cm is made up of 100 turns. It carries current of 5A. Compute magnetic field intensity at the centre of the coil.

$a = 10 \times 10^{-2}\,m$, $N = 100$, $I = 5A$

$$H = \frac{NI}{2a} = \frac{100 \times 5}{2 \times 10^{-2} \times 10}$$

$$\boxed{H = 2500\ AT/m}$$

3) Calculate magnetic flux density due to circular coil of 100 ampere turns and area of $70\ cm^2$ on the axis of a coil at a distance of 10 cm from the centre.

$NI = 100\ AT$, $d = 0.1\,m$

$\pi a^2 = 70\ cm^2$

$\therefore a^2 = 22.28\ cm^2 = 22.28 \times 10^{-4}\ m^2$.

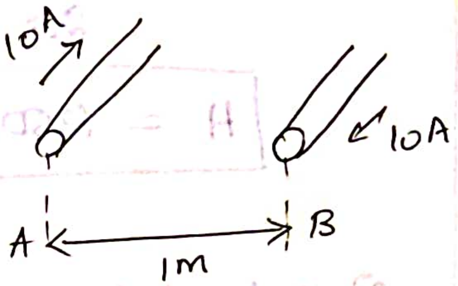
Magnetic flux density $B = \dfrac{\mu_0 NI a^2}{2(a^2+d^2)^{3/2}}$

$$B = \frac{4\pi \times 10^{-7} \times 100 \times 22.28 \times 10^{-4}}{2(22.28 \times 10^{-4} + 0.01)^{3/2}}$$

$$= 103.7 \times 10^{-6}\ \text{Tesla}$$

$$\boxed{B = 103.7\ \mu\,\text{Tesla}}$$

4) A Single phase circuit comprises of two parallel conductors A & B. 1 cm diameter and spaced 1 m apart. The conductors carry current of 10 A & −10 A respectively. Determine field intensity at the surface of each conductor and also in the middle of A + B.



Magnetic field intensity at the surface of B due to A & B is

$$H = \frac{I}{2\pi \cdot 1} + \frac{I}{2\pi \times 0.5 \times 10^{-2}}$$

$$= \frac{10}{2\pi}\left[1 + \frac{10^2}{0.5}\right]$$

$$= \frac{10^3}{\pi} = 318.3 \text{ A/m}.$$

This is same as magnetic field intensity at the surface of A due to A & B.

Magnetic field intensity at mid-point of A+B

$$H = \frac{I}{2\pi(0.5)} + \frac{I}{2\pi(0.5)}$$

$$= \frac{\cancel{2}I}{\cancel{2}\pi(0.5)} = \frac{10}{\pi \times 0.5}$$

$$\boxed{H = 6.366 \text{ A/m}}$$

7) A parallel plate capacitor is composed of tin foil sheets 25 cm × 25 cm on glass as dielectric 6.25 mm thick with $\epsilon_r = 6$. Find capacitance?

$$C = \frac{\epsilon_0 \epsilon_r A}{t} = \frac{8.85\times10^{-12} \times 6 \times .25 \times .25}{6.25\times10^{-3}} = 531\ pf$$

8) A parallel plate capacitor has plates of surface area 0.01 m^2 which are seperated by a distance of 0.005 m in air. Calculate capacitance.

$$C = \frac{\epsilon_0 \epsilon_r A}{t} = \frac{8.85\times10^{-12} \times 1 \times .01}{0.005} = 17.7\ pf$$

9) Evaluate the capacitance of i) Spherical Satellite 1.5 m diameter in free space.
ii) A co-axial cable 1.5 m long filled with polyethylene ($\epsilon_r = 2.26$) with inner conductor of radius 0.6 mm and inner radius of outer conductor 3.5 mm.

i) Capacitance of a single isolated sphere is

$$C = 4\pi\epsilon a$$
$$= 4\pi \times 8.854 \times 10^{-12} \times 0.75 = 83.447\ pf.$$

ii) For co-axial cable,

$$C = \frac{2\pi\epsilon}{\ln(b/a)} L = \frac{2\pi \times 8.85\times10^{-12} \times 2.26 \times 1.5}{\ln\left(\frac{3.5}{0.6}\right)}$$
$$= 106.935\ pf.$$

10). Let $A = 120\ cm^2$, $d = 5\ mm$ and $\epsilon_r = 12$ for a parallel plate capacitor

i) Calculate the capacitance.

ii) After connecting to 40V battery across the capacitor, calculate E, D, Q and the total stored energy.

i) $C = \frac{\epsilon A}{d} = \frac{8.854 \times 10^{-12} \times 12 \times 120 \times 10^{-4}}{5 \times 10^{-3}} = 255\ pf.$

ii). $Q = CV = 255 \times 10^{-12} \times 40 = 10.2\ nC.$

$E = \frac{V}{d} = \frac{40}{5 \times 10^{-3}} = 8\ kV/m$

$D = \epsilon E = 8.85 \times 10^{-12} \times 12 \times 8 \times 10^{3} = 0.85\ \mu C/m^2$

energy $W = \frac{1}{2} CV^2 = \frac{1}{2} \times 255 \times 10^{-12} \times 40^2 = 0.204\ \mu J.$

11) The capacitance of the condenser formed by two parallel metal sheets, each $100\ cm^2$ in area seperated by a dielectric 2mm thick is $2 \times 10^{-4}\ \mu f$. A potential of 20kV is applied to it. Find a) electric flux b) potential gradient in kV/m c) The relative permittivity of the material d) Electric flux density.

Given $A = 100\ cm^2$, $d = 2\ mm$, $C = 2 \times 10^{-4}\ \mu f$
$V = 20\ kV.$

a) $C = \frac{Q}{V}$ (ie) $2 \times 10^{-4} \times 10^{-6} = Q / 20 \times 10^{3}$

$\therefore Q = 4\ \mu C.$

Flux is same as charge.

$$\psi = Q = 4\,\mu C.$$

b) $$E = \frac{V}{d} = \frac{20\times10^{3}}{2\times10^{-3}} = 10\times10^{6}\ V/m = 10\times10^{3}\ kV/m$$

$$= \frac{10\times10^{3}\ kV}{100\,cm} = 100\,kV/cm.$$

c) $$C = \frac{\epsilon_0 \epsilon_r A}{d};$$

$$2\times10^{-4}\times10^{-6} = \frac{8.854\times10^{-12}\times\epsilon_r\times100\times10^{-4}}{2\times10^{-3}}$$

$$\epsilon_r = 4.5177$$

d) $$D = \frac{Q}{A} = \frac{4\times10^{-6}}{100\times10^{-4}} = 4\times10^{-4}\ C/m^2.$$

12) If a potential $V = x^2yz + A\,y^3z$. Find the value of A so that Laplace's equation is satisfied and electric field at $(2, -2, 1)$.

$$V = x^2yz + A\,y^3z.$$

Laplace equation $\nabla^2 V = 0$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = 0$$

$$\frac{\partial v}{\partial x} = 2xyz \quad \text{and} \quad \frac{\partial^2 v}{\partial x^2} = 2yz$$

$$\frac{\partial v}{\partial y} = 3Ay^2z + x^2z \quad \text{and} \quad \frac{\partial^2 v}{\partial y^2} = 6Ayz$$

$$\frac{\partial v}{\partial z} = x^2y + Ay^3 \quad \text{and} \quad \frac{\partial^2 v}{\partial z^2} = 0.$$

$\therefore$ Laplace equation $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = 2yz + 6Ayz$
$= 0$

At $(2,-2,1)$, $2(-2)1 + 6A(-2)1 = 0$

$-4 - 12A = 0$

$A = -1/3$

$\therefore V = x^2yz - \frac{y^3z}{3}$

Electric field $E = -\nabla V$

$= -\left[\frac{\partial v}{\partial x}\hat{a_x} + \frac{\partial v}{\partial y}\hat{a_y} + \frac{\partial v}{\partial z}\hat{a_z}\right]$

$E = -\left[(2xyz)\hat{a_x} + (x^2z - y^2z)\hat{a_y} + (x^2y - \frac{y^3}{3})\hat{a_z}\right]$

$E(2,-2,1) = -\left[2\times2\times(-2)\hat{a_x} + (2^2-(-2)^2)\hat{a_y} + (2^2(-2) - \frac{(-2)^3}{3})\hat{a_z}\right]$

$= 8\hat{a_x} + \frac{16}{3}\hat{a_z}$

13) Calculate the capacitance of a parallel plate capacitor with following details.

Plate area $A = 100\,cm^2$,

Dielectric 1 $\epsilon_{r_1} = 4$, $d_1 = 2mm$

Dielectric 2 $\epsilon_{r_2} = 3$, $d_2 = 3mm$.

If 200V is applied across the plates, what will be the voltage gradient across each dielectric.

Capacitance $C = \frac{A\epsilon_0}{\left[\frac{d_1}{\epsilon_{r1}} + \frac{d_2}{\epsilon_{r2}}\right]}$

UNIT V. PLANE ELECTROMAGNETIC WAVES.

Uniform Plane Wave:

Plane wave → Phase of a wave is same for all points on a plane surface.

Uniform plane wave → Amplitude is constant in a plane wave.

Properties of uniform plane waves:

1. At every point in space electric field (E) & magnetic field (H) are perpendicular to each other and to the direction of travel.
2. The fields vary harmonically with time and at the same frequency, everywhere in space.
3. Each field has the same direction, magnitudes and phase at every point in any plane perpendicular to the direction of wave travel.

Plane Waves in Lossless medium:

Wave equation for free space (Lossless media)

$$\nabla^2 E = \mu\epsilon \frac{\partial^2 E}{\partial t^2}$$

The phasor value of E is $E(x,t) = Re\{E(x)e^{j\omega t}\}$.

According to wave equation,

$$\nabla^2 Re[E e^{j\omega t}] = \mu\epsilon \frac{\partial^2}{\partial t^2} Re[E e^{j\omega t}]$$

$$= \mu\epsilon\, Re[-\omega^2 E e^{j\omega t}]$$

$$Re\{(\nabla^2 E + \mu\epsilon\omega^2 E)e^{j\omega t}\} = 0$$

$$\nabla^2 E + \mu\epsilon\omega^2 E = 0$$

This is wave equation for lossless medium in phasor form & it is called Vector Helmholtz equation

$$\nabla^2 E + \beta^2 E = 0 \quad \text{where } \beta^2 = \mu\epsilon\omega^2.$$

$$\beta = \omega\sqrt{\mu\epsilon} \qquad \alpha = 0$$

Velocity of propagation $\qquad \beta \rightarrow$ Phase shift.

$$v = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu\epsilon}}. \qquad \gamma = \sqrt{j\omega\mu(\sigma + j\omega\epsilon)}$$

lossless $\sigma = 0 \quad \gamma = \pm j\omega\sqrt{\mu\epsilon}$

The wave propagates in x direction (ie) no variation in y & z

$$\frac{\partial^2 E}{\partial x^2} + \beta^2 E = 0 \qquad \sigma = 0 \quad \eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon}} ; \quad \eta = \sqrt{\frac{\mu}{\epsilon}}$$

$$\lambda = \frac{2\pi}{\beta}$$

Solution of the equation is $E = C_1 e^{-j\beta x} + C_2 e^{j\beta x}$.

Plane waves in Lossy medium:

Wave equation for conducting medium is

$$\nabla^2 E - \mu\epsilon\frac{\partial^2 E}{\partial t^2} - \mu\sigma\frac{\partial E}{\partial t} = 0.$$

The phasor wave equation is

$$\nabla^2 E - \mu\epsilon\omega^2 E - j\omega\mu\sigma E = 0.$$

$$\nabla^2 E - j(\omega\mu\sigma + j\mu\epsilon\omega^2)E = 0$$

$$\nabla^2 E - j\omega\mu(\sigma + j\omega\epsilon)E = 0$$

$$\nabla^2 E - \gamma^2 E = 0 \quad \text{where } \gamma^2 = j\omega\mu(\sigma + j\omega\epsilon).$$

$\gamma \rightarrow$ propagation constant which has both real & imaginary parts.

$$\gamma = \alpha + j\beta = \sqrt{j\omega\mu(\sigma + j\omega\epsilon)}$$

where α - attenuation constant

β - phase shift.

$$\gamma^2 = \alpha^2 - \beta^2 + 2j\alpha\beta = j\omega\mu\sigma - \omega^2\mu\epsilon$$

$\therefore$

$$\alpha^2 - \beta^2 = -\omega^2\mu\epsilon$$

$$2\alpha\beta = \omega\mu\sigma$$

To solve these two equations,

$$\alpha^2 + \beta^2 = \sqrt{(\alpha^2 - \beta^2)^2 + 4\alpha^2\beta^2}$$

But, $(\alpha^2 - \beta^2)^2 = (-\omega^2\mu\epsilon)^2$

$$(2\alpha\beta)^2 = (\omega\mu\sigma)^2$$

$$\therefore \alpha^2 + \beta^2 = \sqrt{\omega^4\mu^2\epsilon^2 + \omega^2\mu^2\sigma^2} \quad \text{--- (1)}$$

$$\alpha^2 - \beta^2 = -\omega^2\mu\epsilon \quad \text{--- (2)}$$

(1) + (2)

$$2\alpha^2 = -\omega^2\mu\epsilon + \sqrt{\omega^4\mu^2\epsilon^2 + \omega^2\mu^2\sigma^2}$$

$$\alpha^2 = \frac{-\omega^2\mu\epsilon}{2} + \frac{\omega^2\mu\epsilon}{2}\sqrt{1 + \frac{\sigma^2}{\omega^2\epsilon^2}}$$

$$\therefore \alpha = \sqrt{\frac{\omega^2\mu\epsilon}{2}\left[\sqrt{1 + \frac{\sigma^2}{\omega^2\epsilon^2}} - 1\right]}$$

Attenuation factor $\alpha = \omega\sqrt{\frac{\mu\epsilon}{2}\left[\sqrt{1 + \frac{\sigma^2}{\omega^2\epsilon^2}} - 1\right]}$

From (1) - (2),

$$\beta = \omega\sqrt{\frac{\mu\epsilon}{2}\left[\sqrt{1 + \frac{\sigma^2}{\omega^2\epsilon^2}} + 1\right]}$$

i) Plane waves in low loss dielectrics :

Ratio of conduction current density to displacement current density in medium $= \frac{\sigma}{\omega\epsilon}$

$\frac{\sigma}{\omega\epsilon} = 1 \rightarrow$ dividing line between conductor + dielectrics.

$\frac{\sigma}{\omega\epsilon} >> 1 \rightarrow$ good conductors.

$\frac{\sigma}{\omega\epsilon} << 1 \rightarrow$ good dielectrics.

For dielectrics, $\frac{\sigma}{\omega\epsilon} << 1$.

$$\sqrt{1+\frac{\sigma^2}{\omega^2\epsilon^2}} = \left(1+\frac{\sigma^2}{\omega^2\epsilon^2}\right)^{1/2} = 1+\frac{\sigma^2}{2\omega^2\epsilon^2}$$

Attenuation factor $\alpha = \omega\sqrt{\frac{\mu\epsilon}{2}\left[\sqrt{1+\frac{\sigma^2}{\omega^2\epsilon^2}}-1\right]}$

$$\approx \omega\sqrt{\mu\sigma^2/4\omega^2\epsilon}$$

$$\alpha \approx \frac{\sigma}{2}\sqrt{\frac{\mu}{\epsilon}}$$

Phase shift $\beta \approx \omega\sqrt{\frac{\mu\epsilon}{2}\left[\sqrt{1+\frac{\sigma^2}{\omega^2\epsilon^2}}+1\right]}$

$$\approx \omega\sqrt{\frac{\mu\epsilon}{2}\left[2+\frac{\sigma^2}{2\omega^2\epsilon^2}\right]}$$

$$\approx \omega\sqrt{\mu\epsilon\left[1+\frac{\sigma^2}{4\omega^2\epsilon^2}\right]}$$

$$= \omega\sqrt{\mu\epsilon}\left[1+\frac{\sigma^2}{4\omega^2\epsilon^2}\right]^{1/2}$$

$$\beta \approx \omega\sqrt{\mu\epsilon}\left[1+\frac{\sigma^2}{8\omega^2\epsilon^2}\right]$$

Velocity of wave in dielectric is

$$V = \frac{\omega}{\beta} = \frac{\omega}{\omega\sqrt{\mu\epsilon}\left(1+\frac{\sigma^2}{8\omega^2\epsilon^2}\right)}$$

$$= \frac{1}{\sqrt{\mu\epsilon}}\left(1-\frac{\sigma^2}{8\omega^2\epsilon^2}\right) \qquad \left[\because V_0 = \frac{1}{\sqrt{\mu\epsilon}}\right]$$

$$V = V_0\left(1-\frac{\sigma^2}{8\omega^2\epsilon^2}\right)$$

Intrinsic or characteristic impedance is

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma+j\omega\epsilon}} = \sqrt{\frac{j\omega\mu}{j\omega\epsilon\left(1+\frac{\sigma}{j\omega\epsilon}\right)}}$$

$$\approx \sqrt{\frac{\mu}{\epsilon}\left(1-\frac{\sigma}{j\omega\epsilon}\right)} = \sqrt{\frac{\mu}{\epsilon}}\left(1+\frac{j\sigma}{\omega\epsilon}\right)^{1/2}$$

$$\eta \approx \sqrt{\frac{\mu}{\epsilon}}\left(1+\frac{j\sigma}{2\omega\epsilon}\right)$$

ii) Plane waves in good conductors:

For Good conductor $\frac{\sigma}{\omega\epsilon} >> 1$.

$$\gamma = \sqrt{j\omega\mu(\sigma+j\omega\epsilon)}$$

$$= \sqrt{j\omega\mu\sigma\left(1+\frac{j\omega\epsilon}{\sigma}\right)} \approx \sqrt{j\omega\mu\sigma} \qquad \left[\because \frac{\omega\epsilon}{\sigma} << 1\right]$$

$$\gamma = \sqrt{\omega\mu\sigma}\ \angle 45^\circ$$

$$\therefore \alpha \approx \beta \approx \sqrt{\frac{\omega\mu\sigma}{2}}.$$

Velocity of wave in conductor

$$V = \frac{\omega}{\beta} = \frac{\omega}{\sqrt{\frac{\omega\mu\sigma}{2}}} \approx \sqrt{\frac{2\omega}{\mu\sigma}}$$

Intrinsic impedance of conductor

$$\eta = \sqrt{\frac{j\omega\mu}{j\omega\epsilon\left(1+\frac{\sigma}{j\omega\epsilon}\right)}} \approx \sqrt{\frac{j\omega\mu}{j\omega\epsilon\cdot\frac{\sigma}{j\omega\epsilon}}} \quad \left[\because \frac{\sigma}{\omega\epsilon} \gg 1\right]$$

$$\approx \sqrt{\frac{j\omega\mu}{\sigma}} \approx \sqrt{\frac{\omega\mu}{\sigma}} \angle 45^\circ$$

In good conductors, α and β are large since σ is large. (ie) wave is attenuated greatly as it progresses thro' conductor. But, velocity & characteristic impedance are reduced.

Group Velocity:

The velocity with which the overall shape of a wave amplitude, known as envelope of the wave, propagate thro' a medium is known as group velocity of the wave.

Group velocity is the velocity with which energy propagates.

$$V_g = \frac{\Delta\omega}{\Delta\beta} = \frac{d\omega}{d\beta}.$$

Consider a plane wave propagates in positive x direction $f(t,x) = F_o \cos(\omega t - \beta x)$

If this wave is modulated, group of frequencies centred around carrier frequency is ω.

$$f(x,t) = F_o \cos[(\omega - \Delta\omega)t - (\beta - \Delta\beta)x] +$$
$$F_o \cos[(\omega + \Delta\omega)t - (\beta + \Delta\beta)x]$$

Using trigonometric identity,

$$\cos[(\beta x - \omega t) \pm (\Delta\omega t - \Delta\beta x)] = \cos(\omega t - \beta x)$$
$$\cos(\Delta\omega t - \Delta\beta x) \mp \sin(\omega t - \beta x)\sin(\Delta\omega t - \Delta\beta x).$$

$$\therefore f(x,t) = 2F_o \cos(\omega t - \beta)\cos(\Delta\omega t - \Delta\beta x).$$

Phase Velocity : - Rate at which the phase of the wave propagates in space. This is the speed at which the phase of anyone frequency component of the wave travels.

$$V_p = \frac{\omega}{\beta} = \lambda f = \frac{1}{\sqrt{\mu\epsilon}}$$

Relation between V_p & V_g:

$$V_g = \frac{d\omega}{d\beta} = \frac{d}{d\beta}(\beta V_p) = V_p + \beta \frac{dV_p}{d\beta}.$$

But, $\beta = \frac{2\pi}{\lambda}$, $\therefore \frac{d\beta}{d\lambda} = \frac{-2\pi}{\lambda^2} = -\beta/\lambda$

$$d\beta = -\beta \frac{d\lambda}{\lambda}.$$

$$\therefore \boxed{V_g = V_p - \lambda \frac{dV_p}{d\lambda}}$$

Electromagnetic Power flow & Poynting Vector:

Poynting Vector:

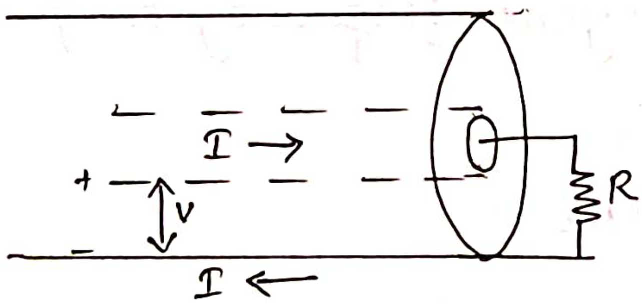
It measures the rate of flow of energy of the wave as it propagates.

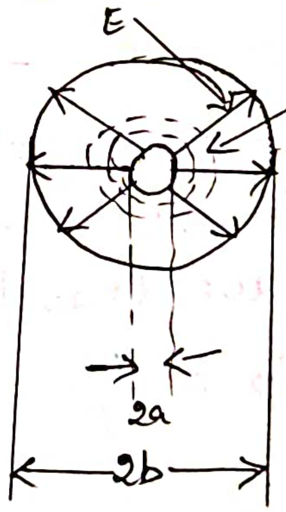
Vector product of electric field intensity and magnetic field intensity is called poynting vector.

$$\boxed{P = E \times H}$$

Power flow in coaxial cable:

Consider a coaxial cable of inner radius 'a' and outer radius 'b' between conductors and steady current I flowing in the inner and outer conductors.





According to Ampere's law, magnetomotive force around any closed circles and the axis of the cable is equal to current enclosed.

$$\oint H.dl = I$$

$$\oint H.dl = H.(2\pi r) \qquad r\text{ – radius of circle}$$

$$\therefore H = I/2\pi r$$

The electric field strength of coaxial cable is

$$E = \frac{V}{r \ln(b/a)}$$

Poynting vector $P = E \times H$.

Since, E & H are perpendicular to each other, magnitude $P = EH$.

The total power flow along the cable is given by integration of the Poynting vector power any cross-sectional surface area $= 2\pi r\, dr$.

$$W = \int_S E \times H \cdot ds = \int_S EH \cdot ds$$

$$= \int_a^b \frac{V}{r \ln(b/a)} \left(\frac{I}{2\pi r}\right) 2\pi r \cdot dr$$

$$= \frac{VI}{\ln(b/a)} \int \frac{dr}{r}$$

$$= \frac{VI}{\ln(b/a)} \left[\ln r\right]_a^b$$

$$= \frac{VI}{\ln(b/a)} \left[\ln b - \ln a\right]$$

$$= \frac{VI}{\ln(b/a)} \ln(b/a)$$

$$W = VI$$

This shows that the power flow along the cable is the product of the voltage and current.

Instantaneous, Average and Complex Poynting vector:

In an ac circuit, the instantaneous power is given by the product of the instantaneous voltage and the instantaneous current.

$$v = Re\,[V e^{j\omega t}] = |V| \cos(\omega t + \theta_v)$$

$$i = Re\,[I e^{j\omega t}] = |I| \cos(\omega t + \theta_i)$$

Instantaneous power is

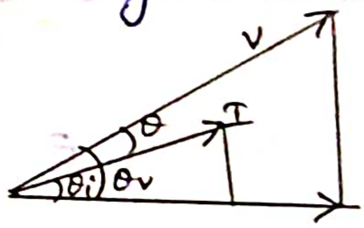
$$W = |V||I| \cos(\omega t + \theta_v) \cos(\omega t + \theta_i)$$

$$= \frac{|V|\,|I|}{2}\left[\cos(\theta_v - \theta_i) + \cos(2\omega t + \theta_v + \theta_i)\right]$$

The instantaneous power flow per square meter (ie) poynting vector is $\bar{P} = \bar{E} \times \bar{H}$.

The average power is $W_{av} = \frac{|V||I|}{2} \cos(\theta_v - \theta_i)$

If $\theta_v - \theta_i = \theta$, the angle between voltage & current, then $W_{av} = \frac{|V||I|}{2} \cos\theta$.



Reactive power $W_{rea} = \frac{|V|\,|I|}{2} \sin\theta$

Complex power $W = \frac{1}{2} V I^*$; I^* – complex conjugate of I.

$$W = \frac{|V|\,|I|}{2} e^{j\theta}.$$

$$W = W_{av} + j\,W_{rea}$$

Complex Poynting Vector $P = \frac{1}{2} E \times H^*$

The real poynting vector (average poynting vector) is

$$P_{av} = \frac{1}{2} Re[E \times H^*]$$

The imaginary poynting vector (reactive poynting vector) is

$$P_{rea} = \frac{1}{2} Im[E \times H^*]$$

In rectangular co-ordinates, the complex Poynting vector normal to y-z plane is

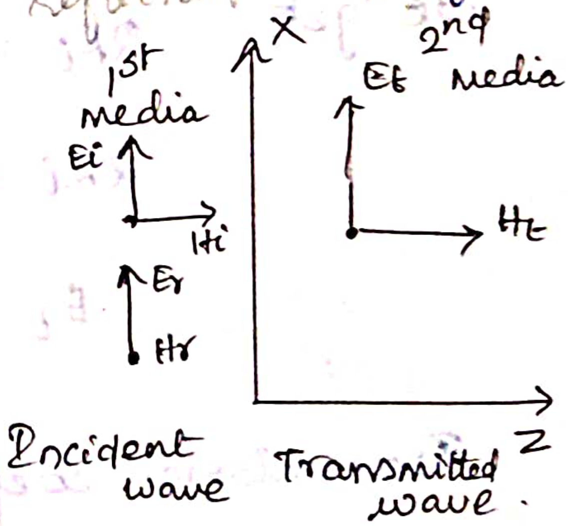
$$P_x = \frac{1}{2}[E_y H_z^* - E_z H_y^*]$$

dielectric
Normal Incidence at a plane Conducting boundary:

Normal incidence of a plane wave → uniform plane wave incidences normally to the boundary between the media.

Consider a uniform plane wave strikes the interface between the two dielectrics at right angles as shown in fig.

Assume that a uniform plane wave travels along +z direction & incidence at right angles at the boundary between two dielectric media at z=0.



Let E_i, E_r → Electric field of incident & reflected wave

E_t → Electric field of transmitted wave.

H_i, H_r → Magnetic field of incident & reflected wave

H_t → Magnetic field of transmitted wave.

The conditions for the total fields in medium 1 are given by, $E_1 = E_i + E_r$, $H_1 = H_i + H_r$.

For medium 2, $E_2 = E_t$, $H_2 = H_t$.

According to boundary conditions, E_t & H_t must be continuous at the interface $z=0$.

$$E_{tan1} = E_{tan2} \quad ; \quad H_{tan1} = H_{tan2} ;$$

At the boundary, (ie) $z=0$

$$E_t = E_i + E_r ; \text{ — (1)} \quad H_t = H_i + H_r .$$

$$E_i = \eta_1 H_i \quad \& \quad E_r = -\eta_1 H_r .$$

$$\therefore \frac{E_i}{\eta_1} - \frac{E_r}{\eta_1} = \frac{E_t}{\eta_2}$$

$$\text{(2) — } E_i - E_r = \frac{\eta_1}{\eta_2} E_t = \frac{\eta_1 (E_i + E_r)}{\eta_2} \rightarrow \text{next pg}$$

From $E_t = E_i + E_r$ and $E_i - E_r = \frac{\eta_1}{\eta_2} E_t$,

we get, (1)+(2) $2E_i = E_t \left(1 + \frac{\eta_1}{\eta_2}\right)$

$$2E_i = E_t \left(\frac{\eta_1 + \eta_2}{\eta_2}\right)$$

$$E_t = \left(\frac{2\eta_2}{\eta_1 + \eta_2}\right) E_i .$$

Transmission coefficient $\tau = \frac{E_t}{E_i} = \frac{2\eta_2}{\eta_1 + \eta_2}$

From $E_i - E_r = \frac{\eta_1}{\eta_2} E_t$, $E_t = \frac{\eta_2}{\eta_1}(E_i - E_r)$ $\quad \frac{2\eta_1}{\eta_1 + \eta_2} \rightarrow$ for H-ny field

$$E_i + E_r = E_t = \frac{\eta_2}{\eta_1}(E_i - E_r)$$

$$\eta_1 (E_i + E_r) = \eta_2 (E_i - E_r)$$

$$E_r(\eta_1 + \eta_2) = E_i(\eta_2 - \eta_1)$$

$$E_r = E_i \left[\frac{\eta_2 - \eta_1}{\eta_1 + \eta_2}\right]$$

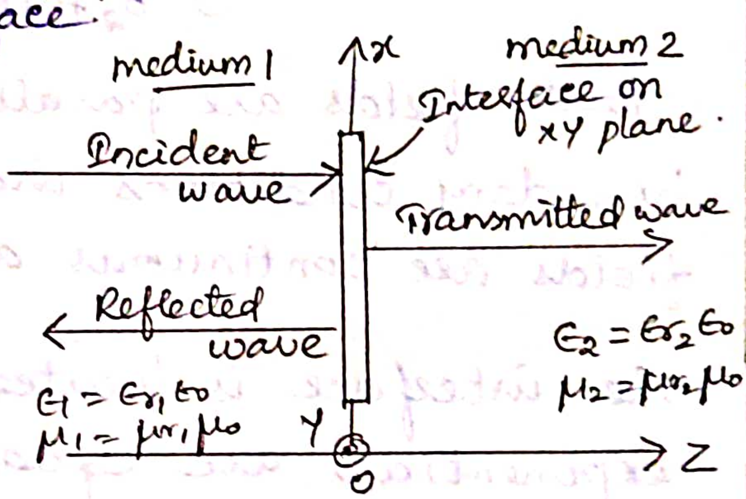
Reflection coefficient $\Gamma = \frac{E_r}{E_i} = \frac{\eta_2 - \eta_1}{\eta_1 + \eta_2}$

cond+ $\frac{\eta_1 - \eta_2}{\eta_1 + \eta_2}$ for mag field

Normal Incidence at a Plane dielectric boundary:

med 1 - die, med 2 - perfect cond. ∴ $\eta_2 = 0$

Consider a plane between two dielectric mediums and a uniform plane wave with normal incidence on the interface.

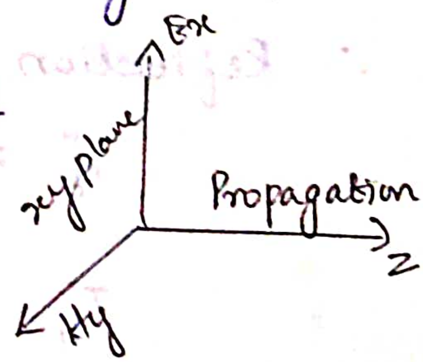


Because of medium discontinuity, incident wave experiences a partial reflection at the interface.

In medium 2, only a forward transmitted wave exists.

The total fields at the interface satisfies the boundary conditions for electromagnetic fields.

The phasor fields in the medium 1 are,



$$E_1(z) = E_1^+ e^{-\gamma_1 z} + E_1^- e^{\gamma_1 z}$$

$$H_1(z) = H_1^+ e^{-\gamma_1 z} + H_1^- e^{\gamma_1 z}$$

$$= \frac{1}{\eta_1}\left[E_1^+ e^{-\gamma_1 z} + E_1^- e^{\gamma_1 z}\right]$$

$$\eta = \frac{E}{H}$$

where $\gamma_1 = \sqrt{j\omega\mu_1(\sigma_1 + j\omega\epsilon_1)}$

$$\eta_1 = \sqrt{\frac{j\omega\mu}{\sigma_1 + j\omega\epsilon_1}}$$

The phasor fields in the medium 2 are,

$$E_2(z) = E_2^+ e^{-\gamma_2 z}$$

$$H_2(z) = H_2^+ e^{-\gamma_2 z} = \frac{1}{\eta_2} E_2^+ e^{-\gamma_2 z}$$

where $\gamma_2 = \sqrt{j\omega\mu_2(\sigma_2 + j\omega\epsilon_2)}$

$$\eta_2 = \sqrt{\frac{j\omega\mu_2}{\sigma_2 + j\omega\epsilon_2}}$$

Both fields are parallel to the interface, the boundary conditions indicate that the total fields are continuous at the interface.

The interface is located at $z=0$ so all exponentials are equal to

$$E_1\big|_{z=0} = E_2\big|_{z=0} \Rightarrow E_1^+ + E_1^- = E_2^+$$

$$H_1\big|_{z=0} = H_2\big|_{z=0} \Rightarrow \frac{1}{\eta_1}(E_1^+ - E_1^-) = \frac{1}{\eta_2} E_2^+$$

$\therefore$ Reflection coefficient is

$$\Gamma_E = \frac{E_1^-}{E_1^+} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \quad ; \text{ for electric field}$$

$$\Gamma_H = \frac{H_1^-}{H_1^+} = \frac{-(E_1^-/\eta_1)}{(E_1^+/\eta_1)} = -\frac{E_1^-}{E_1^+} = -\Gamma_E$$

$$\therefore \Gamma_H = -\Gamma_E = -\left(\frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}\right) ; \text{ for magnetic field.}$$

Problems:

1) In free space, $\bar{E} = 50\cos(\omega t - \beta z)\,\bar{a}_x$ V/m. Find the average power crossing a circular area of radius 2.5 m in the plane $z=0$. Assume $\eta_0 = 120\pi\ \Omega$.

$$\bar{E} = E_m \cos(\omega t - \beta z)\,\hat{a}_N = 50\cos(\omega t - \beta z)\,\bar{a}_x$$

$$\therefore \quad E_m = 50$$

$$E_m = \eta_0 H_m \,; \quad H_m = \frac{E_m}{\eta_0} = \frac{50}{120\pi}$$

In complex form, $\bar{E} = E_m e^{j(\omega t - \beta z)}\,\bar{a}_N$

$$\bar{H} = H_m e^{j(\omega t - \beta z)}\,\bar{a}_N$$

$$\bar{H} = \frac{50}{120\pi} e^{j(\omega t - \beta z)}\,\bar{a}_y$$

$$\bar{H}^* = \frac{50}{120\pi} e^{-j(\omega t - \beta z)}\,\bar{a}_y$$

$$\bar{P}_{avg} = \frac{1}{2}\,Re\,[E \times \bar{H}^*]$$

$$= \frac{1}{2}\,Re\left[\left(50 e^{j(\omega t - \beta z)}\,\bar{a}_x\right)\left(\frac{50}{120\pi} e^{j(\beta z - \omega t)}\,\bar{a}_y\right)\right]$$

$$= \frac{1}{2} \times 50 \times \frac{50}{120\pi} e^{j(\omega t - \beta z)}\,\bar{a}_x \times e^{j(\beta z - \omega t)}\,\bar{a}_y$$

$$= 3.316\,\bar{a}_z \ \ \mathrm{W/m^2}.$$

2) A 300 MHz uniform plane wave propagates through fresh water for which $\sigma = 0$, $\mu = 1$, $\epsilon_r = 78$. Calculate
i) attenuation constant ii) Phase constant
iii) Wavelength iv) Intrinsic impedance.

i) Attenuation constant $\alpha = 0$, since for lossless medium $\sigma = 0$.

ii) Phase constant $\beta = \omega\sqrt{\mu\epsilon}$

$$= 2\pi \times 300 \times 10^6 \sqrt{\mu\epsilon} = 55.53\ \mathrm{rad/m}$$

iii) Wavelength $\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{55.53} = 0.113\,m$

iv) Intrinsic impedance $\eta = \sqrt{\frac{\mu}{\epsilon}} = 42.66\,\Omega$

3) Find frequency after which the earth maybe be considered as perfect dielectric. Assume $\frac{\sigma}{\omega\epsilon} = \frac{1}{100}$. Given $\sigma = 5\times10^{-3}\,S/m$, $\mu_r = 10$ & $\epsilon_r = 8$.

Condition for earth maybe considered as perfect dielectric.

$$\frac{\sigma}{\omega\epsilon} \le \frac{1}{100}$$

$$\therefore \frac{\omega\epsilon}{\sigma} \ge 100$$

$$\frac{\omega\epsilon_0\epsilon_r}{\sigma} \ge 100$$

$$\omega \ge \frac{100\times\sigma}{\epsilon_0\epsilon_r}$$

$$2\pi f \ge \frac{100\times5\times10^{-3}}{8.854\times10^{-12}\times8\times2\pi}$$

$$f \ge 1.12\times10^{9}$$

$$f = 1.102\,GHz.$$

4) The electric field associated with a plane wave travelling in a perfect dielectric medium is given by, $E_x(z,t) = 10\cos(2\pi\times10^7 t - 0.1\pi x)$ V/m. Find the velocity of propagation and intrinsic impedance. $[\mu_r = 1]$

$$E_x(z,t) = 10\cos(2\pi\times10^7 t - 0.1\pi x) \text{ V/m}$$

$$= E_m \cos(\omega t - \beta x) \text{ V/m}$$

$\therefore$ $E_m = 10$; $\omega = 2\pi\times10^7$ rad/sec ; $\beta = 0.1\pi$

i) Velocity of propagation $v = \frac{1}{\sqrt{\mu\epsilon}}$; $\left[\because c = \frac{1}{\sqrt{\mu_0\epsilon_0}}\right]$

$$\therefore \quad v = \frac{c}{\sqrt{\mu_r\epsilon_r}} = \frac{c}{\sqrt{\epsilon_r}} \quad [\because \mu_r = 1]$$

To find ϵ_r : $\beta = \omega\sqrt{\mu\epsilon} = \omega\sqrt{\mu_0\epsilon_0\epsilon_r}$

$$0.1\pi = 2\pi\times10^{-7}\sqrt{4\pi\times10^{-7}\times8.854\times10^{-12}\times\epsilon_r}$$

$$\epsilon_r = 2.25$$

$$\therefore \quad v = \frac{3\times10^8}{\sqrt{2.25}} = 2\times10^8 \text{ m/s}$$

$[\eta_0 = 377]$

ii) Intrinsic impedance $\eta = \eta_0\sqrt{\frac{\mu_r}{\epsilon_r}} = 251.5\,\Omega$

5) A uniform plane wave is travelling at a velocity of 2.5×10^5 m/s having wavelength 0.25 mm in a non-magnetic good conduct. Calculate the frequency of wave and the conductivity of the medium.

$$v = 2.5\times10^5 \text{ m/s} \; ; \; \lambda = 0.25\times10^{-3} \text{ m}$$

$$f = \frac{v}{\lambda} = 1\times10^9 \text{ Hz}$$

$$\beta = \sqrt{\pi f\mu\sigma} \, , \quad \sigma = \frac{\beta^2}{\pi f\mu}$$

$V = \frac{\omega}{\beta}$; $\therefore \beta = \frac{\omega}{V} = \frac{2\pi f}{V} = 25.13 \times 10^3$ rad/m.

$$\sigma = \frac{25.13^2 \times 10^6}{\pi \times 1 \times 10^9 \times 4\pi \times 10^{-7}} = 1.6 \times 10^5 \text{ S/m}$$

6) A uniform plane wave in freespace is normally incident on a dielectric having $\epsilon_r = 4$, $\mu_r = 1$. Electric field of incident wave is $\bar{E}_x = E_0 e^{-jz} \bar{a}_x$ to $z<0$ where E_0 is real constant. Calculate f, λ of incident, reflected wave and magnetic field of incident wave.

$\bar{E}_x = E_0 e^{-j\beta z} \bar{a}_x$ V/m $\Rightarrow \beta = 1$ rad/m.

$\lambda = 2\pi/\beta = 6.28$ m

$V = f\lambda$; $f = 47.7 \times 10^6$ Hz

$\bar{H}_i = H_0 e^{-jz} \bar{a}_y$

$H_0 = \frac{E_0}{\eta_0} = 2.65 \times 10^{-3} E_0$ $[\eta_0 = 377\Omega]$

$\bar{H}_i = 2.65 \times 10^{-3} E_0 e^{-jz} \bar{a}_y$ A/m.

7) $\bar{E}$ and $\bar{H}$ waves travelling in free space, are normally incident on the inteface with a perfect dielectric with $\epsilon_r = 3$. Compute the magnitudes of incident, reflected & transmitted E & H waves at the interface.

Medium 1 is a free space : $\eta_1 = 377\Omega$.

Medium 2 is dielectric: $\eta_2 = \sqrt{\frac{\mu}{\epsilon}} = 217.51\ \Omega$.

Transmission coefficient $\tau = \frac{E_t}{E_i} = \frac{2\eta_2}{\eta_1 + \eta_2}$
$= 0.732$.

$\frac{E_t}{E_i} = 0.732 \Rightarrow E_t = 0.732\, E_i$ V/m

It is the magnitude of transmitted E wave.

The reflection coefficient $\Gamma = \frac{E_r}{E_i} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$

$\Gamma = -0.268$

$\frac{E_r}{E_i} = -0.268 \Rightarrow E_r = -0.268\, E_i$ V/m

It is the magnitude of reflected E-Wave.

For magnitude fields,

$$\frac{H_t}{H_i} = \frac{\eta_1}{\eta_2}\left(\frac{E_t}{E_i}\right) = \frac{377}{217.51}(0.732) = 1.268$$

$H_t = 1.268\, H_i$ A/m $\rightarrow$ Magnitude of transmitted H-wave.

$$\frac{H_r}{H_i} = -\frac{E_r}{E_i} = 0.268$$

$\therefore H_r = 0.268\, H_i$ A/m $\rightarrow$ Magnitude of reflected H-wave.

8) The phase velocity in a material is $\sqrt{g/k}$ where k is propagation constant. P.T. group velocity will be half of phase velocity.

Phase Velocity $V_p = \frac{\omega}{k} = \sqrt{g/k}$

$\omega^2 = g \cdot k$ —— ①

Group velocity $V_g = \frac{d\omega}{dk}$ —— ②

From ①, $2\omega \frac{d\omega}{dk} = g \Rightarrow \frac{d\omega}{dk} = \frac{g}{2\omega}$ —— ③

Substitute ③ in ②, $V_g = g/2\omega$

From ①, $\omega = \sqrt{gk} \Rightarrow V_g = \frac{g}{2\sqrt{gk}} = \frac{1}{2}\sqrt{g/k} = \frac{1}{2} V_p$.

9) Find the velocity of a plane wave in a lossless medium having $\epsilon_r = 4$ and $\mu_r = 1.2$.

$$V = \frac{1}{\sqrt{\mu\epsilon}} = \frac{3\times10^8}{\sqrt{1.2\times4}} = 1.37\times10^8 \ m/s$$

$$\left[\because V_0 = \frac{1}{\sqrt{\mu_0\epsilon_0}} = 3\times10^8 m/s\right]$$

10) Find the characteristic impedance of the medium whose $\epsilon_r = 3$, $\mu_r = 1$.

$$\eta = \sqrt{\frac{\mu}{\epsilon}} = \sqrt{\frac{\mu_0\mu_r}{\epsilon_0\epsilon_r}} = \eta_0\sqrt{\frac{\mu_r}{\epsilon_r}}$$

$$= 120\pi\sqrt{\frac{1}{3}} = 217.66\ \Omega.$$

11) The electric flux intensity of plane wave travelling in a perfect dielectric medium is $E_x = 10\cos(2\pi\times10^7 t - 0.1\pi z)$ V/m. Find the expression for magnetic field intensity if $\mu_r = 1$.

$$E_x = A\cos(\omega t - \beta z)$$

$$= 10\cos(2\pi\times10^7 t - 0.1\pi z)$$

$\therefore \omega = 2\pi\times10^7$, $\beta = 0.1\pi$

$2\pi f = 10^7\times2\pi$ $\qquad \frac{2\pi}{\lambda} = 0.1\pi$

$f = 10^7$ Hz $\qquad \lambda = 20$ m

Velocity of propagation in a medium $v = f\lambda = 10^7 \times 20$

$= 2 \times 10^8$ m/s.

$$\frac{\text{Velocity in free space}}{\text{Velocity in a medium}} = \frac{\sqrt{\mu_0 \mu_r \epsilon_0 \epsilon_r}}{\sqrt{\mu_0 \epsilon_0}} = \sqrt{\mu_r \epsilon_r}$$

$$\sqrt{\mu_r \epsilon_r} = \frac{3 \times 10^8}{2 \times 10^8} = 1.5$$

$$\sqrt{\epsilon_r} = 1.5. \quad \therefore \epsilon_r = 2.25 \quad [\mu_r = 1]$$

Intrinsic impedance $\eta = \sqrt{\frac{\mu}{\epsilon}} = \sqrt{\frac{\mu_0 \mu_r}{\epsilon_0 \epsilon_r}}$

$$= \frac{1}{\sqrt{\epsilon_r}} \sqrt{\frac{\mu_0}{\epsilon_0}} = \frac{120\pi}{1.5} = 80\pi$$

The magnetic field intencity in z-direction,

$$H_z = \frac{E_x}{\eta} = \frac{10}{80\pi} \cos(2\pi \times 10^7 t - 0.1\pi z) \text{ A/m}$$

$$= 0.04 \cos(2\pi \times 10^7 t - 0.1\pi z) \text{ A/m}.$$